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Fourier's Constants of Functions of Several Variables.

BY DR. W. W. KÜSTERMANN.

In Vol. LVII of the *Mathematische Annalen*, A. Hurwitz has shown how to express the product of two ordinary Fourier's series in form of another Fourier's series, or, stated differently, how to compute Fourier's constants of the product of two functions from those of the functions themselves. In the following pages we solve the corresponding problem for multiple Fourier's series. In order to simplify matters the work is carried through for double Fourier's series only, but results and proofs admit of immediate and obvious generalization to n -tuple series.

The solution of our problem depends vitally upon a proof of the relation*

$$\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy = \sum_{\mu, \nu=0}^{\infty} \frac{(a^2 + b^2 + c^2 + d^2)_{\mu\nu}}{2^{E(\frac{1}{\mu+1}) + E(\frac{1}{\nu+1})}} \quad (1)$$

where $(a, b, c, d)_{\mu\nu}$ are Fourier's constants of $f(x, y)$ and have the values

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}_{\mu\nu} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\alpha, \beta) \begin{Bmatrix} \cos \mu\alpha \cos \nu\beta \\ \sin \mu\alpha \cos \nu\beta \\ \cos \mu\alpha \sin \nu\beta \\ \sin \mu\alpha \sin \nu\beta \end{Bmatrix} d\alpha d\beta. \quad (2)$$

The analogue of this relation for functions of a single variable is due to Parseval and was proved by him on the supposition that the function in question is really represented by its Fourier's development and that this series is integrable term by term. In 1893 de la Vallée-Poussin† gave a proof requiring merely that the function and its square be integrable. Hurwitz, in 1903, again called attention to the wide scope of the theorem, gave a new proof, and applied

* In the following $(a^2 + b^2 + c^2 + d^2)_{\mu\nu}$ stands for $(a^2 + b^2)_{\mu\nu} + (c^2 + d^2)_{\mu\nu}$ and so on. $E(z)$ denotes the largest positive integer contained in z . Hereafter we will write $\Delta_{\mu\nu}$ in place of $2^{E(\frac{1}{\mu+1}) + E(\frac{1}{\nu+1})}$,

then $\Delta_{\mu\nu} = \begin{cases} 4 & \text{when } \mu = \nu = 0, \\ 2 & \text{" } \mu = 0, \nu > 0 \text{ or } \mu > 0, \nu = 0, \\ 1 & \text{" } \mu > 0, \nu > 0. \end{cases}$

† De la Vallée-Poussin, *Ann. de la Soc. Scient. de Bruxelles*, Vol. XVII, p. 18.

it to the problem mentioned. More recently proofs of varying generality have been added by Fischer, Stekloff, Fatou,* and the formula has gained considerably in interest through the researches of Riess and Fischer (Riess-Fischer Theorem).†

The first three paragraphs of our paper are devoted to proving relation (1) by a method which like those of de la Vallée-Poussin, Fischer and Fatou involves the use of Poisson's integral. (For a brief survey see § 3.) We, too, make no assumption regarding the convergence of the Fourier's series of $f(x, y)$.

All we demand of $f(x, y)$ is that it be bounded and possess a Riemann double integral over the fundamental square.‡ For convenience of language we will suppose $f(x, y)$ periodic 2π , so that

$$f(x \pm 2\pi k, y \pm 2\pi l) = f(x, y) \text{ for } \begin{cases} k=0, 1, 2, \dots, \\ l=0, 1, 2, \dots \end{cases}$$

These conditions insure the existence of Fourier's constants $(a, b, c, d)_{\mu\nu}$, but do not in any way prejudice the question of convergence§ of the formal Fourier's development for $f(x, y)$ which we can write:

$$f(x, y) \sim \sum_{\mu, \nu=0}^{\infty} \frac{1}{\Lambda_{\mu\nu}} [(a_{\mu\nu} \cos \mu x + b_{\mu\nu} \sin \mu x) \cos \nu y + (c_{\mu\nu} \cos \mu x + d_{\mu\nu} \sin \mu x) \sin \nu y]. \quad (3)$$

How relation (1) can be utilized to solve the problem originally proposed is shown in § 4.

§ 1. Preliminary Theorems.

THEOREM I. *If $\phi(x, y)$ is bounded and integrable|| in the fundamental square and can be represented by a trigonometric double series which is uniformly convergent in this closed region, then the trigonometric series is a Fourier's series (that is its coefficients are given by formulas of type (2)).*

The proof is so closely analogous to the proof of the corresponding theorem for ordinary Fourier's series that we will not reproduce it here.

* Fischer, *Monatshefte*, Vol. XV, p. 69; Stekloff, *Mem. de l'Ac. de St. Petersburg*, Vol. XV, Ser. 8; Fatou, *Acta Math.*, Vol. XXX.

† Riess, "Über orthogonale Funktionensysteme," *Gött. Nachr.*, 1907; Fischer, *Compt. Rend.*, 1907; also Young, *Quart. Journ.*, Vol. XLIV; where this relation is shown to represent the necessary and sufficient condition that a given trigonometric series be the Fourier's series of a function whose square is integrable.

‡ By the "fundamental square" we mean the square $(-\pi, -\pi; \pi, \pi)$ whose lower left-hand and upper right-hand vertices are $(-\pi, -\pi)$ and (π, π) , respectively.

§ In the Stolz-Pringsheim sense. Cf. *Münchener Sitzungsber.*, Vol. XXVII, p. 101.

|| All integrals in this paper are to be taken in the Riemann sense.

THEOREM II. If $f(x, y)$ and $\phi(x, y)$ are both bounded and integrable in the fundamental square, if furthermore $\phi(x, y)$ can be developed into a double Fourier's series, uniformly convergent in this region, then

$$\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \phi(x, y) dx dy = \sum_{\mu, \nu=0}^{\infty} (a\alpha + b\beta + c\gamma + d\delta)_{\mu\nu} / \Lambda_{\mu\nu},^* \quad (4)$$

where $(a, b, c, d)_{\mu\nu}$, $(\alpha, \beta, \gamma, \delta)_{\mu\nu}$ are the Fourier's constants of f and ϕ , respectively.

To prove this theorem multiply the Fourier's expansion of $\phi(x, y)$ by $f(x, y)$ and integrate term by term over the fundamental square.† If, in particular, we let $\phi(x, y) = f(x, y)$, equation (4) goes over into (1). The latter relation therefore holds in the special case, where $f(x, y)$ can be developed into a uniformly convergent Fourier's series. To prove it for any integrable $f(x, y)$ we need further preparations.

LEMMA: The double series

$$S = \sum_{\mu, \nu=0}^{\infty} (a^2 + b^2 + c^2 + d^2)_{\mu\nu} / \Lambda_{\mu\nu}, \quad (5)$$

where $(a, b, c, d)_{\mu\nu}$ are the Fourier's constants of a bounded integrable function $f(x, y)$, is absolutely convergent and can be summed by rows, columns or diagonals.

PROOF: Define $\psi(x, y)$ by the identity

$$\begin{aligned} \psi(x, y) = f(x, y) - & \left\{ \frac{a_{00}}{4} + \frac{1}{2} \sum_{\mu=1}^m (a_{\mu 0} \cos \mu x + b_{\mu 0} \sin \mu x) \right. \\ & + \frac{1}{2} \sum_{\nu=1}^n (a_{0\nu} \cos \nu y + c_{0\nu} \sin \nu y) \\ & + \sum_{\mu'=1}^m \sum_{\nu'=1}^n [(a_{\mu'\nu'} \cos \mu' x + b_{\mu'\nu'} \sin \mu' x) \cos \nu' y \\ & \left. + (c_{\mu'\nu'} \cos \mu' x + d_{\mu'\nu'} \sin \mu' x) \sin \nu' y] \right\}. \end{aligned}$$

* Cf. note * p. 113.

† The reduction is effected by means of the well-known formulas:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin \kappa x}{\cos \kappa x} \cdot \frac{\sin \lambda x}{\cos \lambda x} dx &= \begin{cases} 0, & \text{if } \kappa \neq \lambda, \\ \pi, & \text{if } \kappa = \lambda \neq 0. \end{cases} \\ \int_{-\pi}^{\pi} \sin \kappa x \cdot \cos \lambda x dx &= 0; \quad \int_{-\pi}^{\pi} \frac{\sin \kappa x}{\cos \kappa x} dx = 0, & \text{if } \kappa \neq 0. \end{aligned}$$

Squaring both sides of this equation

$$\begin{aligned}
 [\psi(x, y)]^2 = [f(x, y)]^2 - & \left\{ \frac{a_{00}}{2} f(x, y) + \sum_{\mu=1}^m (a_{\mu 0} f(x, y) \cos \mu x + b_{\mu 0} f(x, y) \sin \mu x) \right. \\
 & + \sum_{\nu=1}^n (a_{0\nu} f(x, y) \cos \nu y + c_{0\nu} f(x, y) \sin \nu y) \\
 & + 2 \sum_{\mu'=1}^m \sum_{\nu'=1}^n [a_{\mu' \nu'} f(x, y) \cos \mu' x \cos \nu' y + b_{\mu' \nu'} f(x, y) \sin \mu' x \cos \nu' y \\
 & \quad + c_{\mu' \nu'} f(x, y) \cos \mu' x \sin \nu' y + d_{\mu' \nu'} f(x, y) \sin \mu' x \sin \nu' y] \Big\} \quad (6) \\
 & + \frac{a_{00}^2}{16} + \frac{1}{4} \sum_{\mu=1}^m (a_{\mu 0}^2 \cos^2 \mu x + b_{\mu 0}^2 \sin^2 \mu x) + \frac{1}{4} \sum_{\nu=1}^n (a_{0\nu}^2 \cos^2 \nu y + c_{0\nu}^2 \sin^2 \nu y) \\
 & + \sum_{\mu'=1}^m \sum_{\nu'=1}^n [a_{\mu' \nu'}^2 \cos^2 \mu' x \cos^2 \nu' y + b_{\mu' \nu'}^2 \sin^2 \mu' x \cos^2 \nu' y \\
 & \quad + c_{\mu' \nu'}^2 \cos^2 \mu' x \sin^2 \nu' y + d_{\mu' \nu'}^2 \sin^2 \mu' x \sin^2 \nu' y] + \Sigma \mu \Sigma \nu \Sigma \mu' \Sigma \nu'.
 \end{aligned}$$

Here $\Sigma \mu \Sigma \nu \Sigma \mu' \Sigma \nu'$ contains those product terms (except for constant coefficients) which are of one of the sixteen types $\frac{\cos \mu x}{\sin \mu x} \cdot \frac{\cos \nu y}{\sin \nu y} \cdot \frac{\cos \mu' x}{\sin \mu' x} \cdot \frac{\cos \nu' y}{\sin \nu' y}$ where some of the numbers μ, ν, μ', ν' may be zero, but in no term of which the equalities $\mu = \mu', \nu = \nu'$ coexist. If we now integrate equation (6) over the fundamental square the terms resulting from $\Sigma \mu \Sigma \nu \Sigma \mu' \Sigma \nu'$ vanish* and we have

$$\begin{aligned}
 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [\psi(x, y)]^2 dx dy = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy - \pi^2 \Big\{ \frac{a_{00}^2}{2} + \sum_{\mu=1}^m (a^2 + b^2)_{\mu 0} \\
 + \sum_{\nu=1}^n (a^2 + c^2)_{0\nu} + 2 \sum_{\mu=1}^m \sum_{\nu=1}^n (a^2 + b^2 + c^2 + d^2)_{\mu\nu} \Big\} \\
 + \pi^2 \Big\{ \frac{a_{00}^2}{4} + \frac{1}{2} \sum_{\mu=1}^m (a^2 + b^2)_{\mu 0} + \frac{1}{2} \sum_{\nu=1}^n (a^2 + c^2)_{0\nu} \\
 + \sum_{\mu=1}^m \sum_{\nu=1}^n (a^2 + b^2 + c^2 + d^2)_{\mu\nu} \Big\}
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [\psi(x, y)]^2 dx dy \\
 = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy - \sum_{\substack{\mu=m, \\ \nu=0}}^{\mu=m, \nu=n} (a^2 + b^2 + c^2 + d^2)_{\mu\nu} / \Lambda_{\mu\nu}.
 \end{aligned}$$

The integral on the left-hand side can clearly not be negative, hence, setting

$$S_{mn} = \sum_{\mu=m, \nu=n}^{\mu=m, \nu=n} (a^2 + b^2 + c^2 + d^2)_{\mu\nu} / \Lambda_{\mu\nu}$$

we can write

$$S_{mn} \leq \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy,$$

* Cf. note † p. 115.

no matter what positive integral values m and n may have. Since, moreover, all the terms of (5) are positive, it follows from a well-known theorem* that this series converges absolutely and hence can be summed in the various ways mentioned.

COROLLARY: *The Fourier's constants of a bounded and integrable function of two variables approach zero when either one or both subscripts become infinite.*†

§ 2. Discussion of Poisson's Double Integral.

If in the double Fourier's series (3) we introduce the convergence factors $r^{\mu+\nu}$, where $0 \leq r < 1$, we obtain a new double series

$$u_r(x, y) = \sum_{\mu, \nu=0}^{\infty} \frac{r^{\mu+\nu}}{\Lambda_{\mu\nu}} [(a_{\mu\nu} \cos \mu x + b_{\mu\nu} \sin \mu x) \cos \nu y + (c_{\mu\nu} \cos \mu x + d_{\mu\nu} \sin \mu x) \sin \nu y] \quad (7)$$

which is uniformly (and absolutely) convergent for all values of (x, y) and hence defines a function $u_r(x, y)$ which is finite, continuous, and integrable in the fundamental square. If we replace $(a, b, c, d)_{\mu\nu}$ by their values from (2), interchange summation and integration signs, and effect the summation by means of the formula

$$\frac{1}{2} + \sum_{\mu=1}^{\infty} r^{\mu} \cos \mu \phi = \frac{1}{2} \cdot \frac{1-r^2}{1+r^2-2r \cos \phi} = \frac{1}{2} P_r(\phi),$$

we obtain

$$u_r(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\alpha, \beta) P_r(\alpha-x) P_r(\beta-y) d\alpha d\beta,$$

which by the transformation $\alpha-x=\xi$, $\beta-y=\eta$ goes over into

$$\begin{aligned} u_r(x, y) &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+\xi, y+\eta) P_r(\xi) P_r(\eta) d\xi d\eta \\ &= \frac{1}{4\pi^2} \int_0^{\pi} \int_0^{\pi} F(x, y, \xi, \eta) P_r(\xi) P_r(\eta) d\xi d\eta, \end{aligned} \quad (8)$$

where

$$F(x, y, \xi, \eta) \equiv f(x+\xi, y+\eta) + f(x+\xi, y-\eta) + f(x-\xi, y+\eta) + f(x-\xi, y-\eta). \quad (9)$$

* Cf. Pringsheim, *loc. cit.*, § 2, V and § 3.

† This property of double Fourier's constants was proved from the integrals defining them by W. H. Young, *Lond. Math. Soc. Proc.*, Ser. 2, Vol. XI, p. 133.

We wish to prove

THEOREM III. *If $f(x, y)$ is bounded and integrable in the fundamental square, then*

$$\lim_{r \rightarrow 1-0} u_r(x, y) = f(x, y)$$

at every point of continuity of $f(x, y)$.

Since

$$\int_0^s P_r(\phi) d\phi = 2 \arctan \left(\frac{1+r}{1-r} \tan \frac{s}{2} \right) \leq \pi$$

and

$$\int_0^\pi P_r(\phi) d\phi = \pi \text{ for every positive } r < 1,$$

we may write

$$\left. \begin{aligned} u_r(x, y) - f(x, y) &= \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi [F(x, y, \xi, \eta) - 4f(x, y)] P_r(\xi) P_r(\eta) d\xi d\eta \\ &= \frac{1}{4\pi^2} \left\{ \int_0^a \int_0^a + \int_a^\pi \int_0^\pi + \int_0^a \int_a^\pi \right\} \\ &\quad [F(x, y, \xi, \eta) - 4f(x, y)] P_r(\xi) P_r(\eta) d\xi d\eta \\ &= \frac{1}{4\pi^2} \{ I_1 + I_2 + I_3 \}. \end{aligned} \right\} \quad (10)$$

We shall see that the sum of these three integrals can be made as small as we please by choosing r close enough to unity.

$$|I_1| \leq \int_0^a \int_0^a |F(x, y, \xi, \eta) - 4f(x, y)| \cdot |P_r(\xi)| \cdot |P_r(\eta)| d\xi d\eta.$$

Now

$$P_r(\phi) = \frac{1-r^2}{1+r^2-2r\cos\phi} \geq \frac{1-r^2}{(1+r)^2} = \frac{1-r}{1+r} > 0$$

for $0 \leq r < 1$; hence, $|P_r(\phi)| = P_r(\phi)$; furthermore,

$$1+r^2-2r\cos\phi = (1-r)^2 + 4r\sin^2\frac{\phi}{2} > 4r\sin^2\frac{a}{2},$$

for $0 \leq r < 1$ and $0 < a \leq \phi \leq \pi$; therefore,

$$P_r(\phi) < (1-r^2)/4r\sin^2\frac{a}{2}.$$

If (x, y) is a point of continuity for $f(x, y)$, then a can be chosen so small that the upper bound of $|F(x, y, \xi, \eta) - 4f(x, y)|$ in the domain $\begin{cases} 0 \leq \xi \leq a \\ 0 \leq \eta \leq a \end{cases}$ is less than $\frac{\varepsilon_1}{\pi^2}$. Thus,

$$|I_1| < \varepsilon_1 \text{ for any } 0 \leq r < 1. \quad (11)$$

Denoting now by M the upper bound of $|f(x, y)|$ in the fundamental square, we get

$$|I_2 + I_3| \leq 8M \left| \int_a^\pi P_r(\xi) d\xi \int_0^\pi P_r(\eta) d\eta + \int_0^a P_r(\xi) d\xi \int_a^\pi P_r(\eta) d\eta \right| \\ \leq 4\pi M(1-r^2)/r \sin^2 \frac{a}{2}, \quad (12)$$

a having been fixed so as to insure (11) we may choose $R < 1$ so close to unity that the right-hand side of (12) is less than ϵ_2 for $R \leq r < 1$; hence,

$$|I_1| + |I_2 + I_3| < \epsilon_1 + \epsilon_2, \text{ for } R \leq r < 1.$$

It follows from (10) that $\lim_{r \rightarrow 1-0} u_r(x, y) = f(x, y)$.

COROLLARY: If the bounded function $f(x, y)$ is integrable (according to Riemann's definition) in the fundamental square and $u_r(x, y)$ is defined by series (7), then

$$\lim_{r \rightarrow 1-0} \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [u_r(x, y)]^2 dx dy = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy. \quad (13)$$

PROOF: Since $f(x, y)$ is integrable in the sense of Riemann its points of discontinuity form a set of measure zero.* Consequently, $[u_r(x, y)]^2$ approaches $[f(x, y)]^2$ "almost everywhere."† Moreover, $u_r(x, y)$ is easily shown to be bounded in the aggregate for which $0 \leq r < 1$.‡ Hence, we can apply a theorem of Lebesgue§ which states that if a sequence of integrable functions, bounded in absolute value, converges almost everywhere to a limiting function which is integrable, then the limit of the corresponding sequence of integrals is equal to the integral of the limiting function. It follows that (13) is true if the integrals involved are interpreted as Lebesgue integrals; but since, in view of our hypothesis, all these integrals have a meaning also according to Riemann's definition, and since the two kinds of integrals are identical, whenever they both exist, our corollary is proved for Riemann integrals. ||

* Cf. Hobson, "Theory of Functions of a Real Variable," § 311, p. 419.

† I. e., except at a set of measure zero.

‡ $|u_r(x, y)| \leq \frac{1}{4\pi^2} \cdot 4M \cdot \int_0^\pi P_r(\xi) d\xi \int_0^\pi P_r(\eta) d\eta \leq M$ for $0 \leq r < 1$.

§ Lebesgue, *Ann. de l'École Normale*, Ser. 3, Vol. XXVII (1910), p. 375.

|| This method is no longer applicable when $f(x, y)$ is only known to be integrable in Lebesgue's sense, for then the points of discontinuity may form a set of measure greater than zero.

§ 3. *Proof of the Generalized Parseval's Formula.*

We are now in a position to prove relation (1). Since the trigonometric double series

$$u_r(x, y) = \sum_{\mu, \nu=0}^{\infty} \frac{r^{\mu+\nu}}{\Lambda_{\mu\nu}} [(a_{\mu\nu} \cos \mu x + b_{\mu\nu} \sin \mu x) \cos \nu y + (c_{\mu\nu} \cos \mu x + d_{\mu\nu} \sin \mu x) \sin \nu y] \quad (7)$$

is uniformly convergent we see from Theorem I, § 1, that it is the Fourier's development of $u_r(x, y)$ and that the Fourier's constants of $u_r(x, y)$ are $r^{\mu+\nu}(a, b, c, d)_{\mu\nu}$. Hence, Theorem II is applicable and we have

$$\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [u_r(x, y)]^2 dx dy = \sum_{\mu, \nu=0}^{\infty} r^{2(\mu+\nu)} (a^2 + b^2 + c^2 + d^2)_{\mu\nu} / \Lambda_{\mu\nu}. \quad (14)$$

Now by the corollary of § 2,

$$\begin{aligned} \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(x, y)]^2 dx dy &= \lim_{r=1-0} \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [u_r(x, y)]^2 dx dy \\ &= \lim_{r=1-0} \sum_{\mu, \nu=0}^{\infty} r^{2(\mu+\nu)} (a^2 + b^2 + c^2 + d^2)_{\mu\nu} / \Lambda_{\mu\nu}. \end{aligned} \quad (15)$$

The latter series of positive terms converges uniformly in r for the values $0 \leq r \leq 1$, since it is term by term less than or equal to the series of positive constant terms (5) known to be convergent by the lemma of § 1. Hence, we may proceed to the limit $r=1-0$ in each term of (15) separately which gives (1).

§ 4. *Fourier's Constants of the Product $f(x, y) \cdot \phi(x, y)$.*

It remains to be shown how relation (1) may be used to solve the problem originally proposed, to compute the Fourier's constants $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})_{\mu\nu}$ of $f \cdot \phi$, having given those of f and ϕ , which are $(a, b, c, d)_{\mu\nu}$ and $(\alpha, \beta, \gamma, \delta)_{\mu\nu}$, respectively. Since $f \cdot \phi = \frac{1}{4}[f+\phi]^2 - \frac{1}{4}[f-\phi]^2$ and the Fourier's constants of $f \pm \phi$ are $(a \pm \alpha, b \pm \beta, c \pm \gamma, d \pm \delta)_{\mu\nu}$ we obtain from (1)

$$\mathfrak{A}_{00} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \cdot \phi dx dy = \sum_{\mu, \nu=0}^{\infty} (a\alpha + b\beta + c\gamma + d\delta)_{\mu\nu} / \Lambda_{\mu\nu}. \quad (16)$$

To find $\mathfrak{A}_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \cdot \phi \cos mx \cos ny dx dy$ we place for the moment

$$\psi = \phi \cos mx \cos ny.$$

The Fourier's constants of ψ , namely $(A', B', C', D')_{\mu\nu}$, can be expressed in terms of those of ϕ . For instance,

$$\begin{aligned} A'_{\mu\nu} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \psi' \cos \mu x \cos \nu y \, dx dy = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi \cos m x \cos \mu x \cos n y \cos \nu y \, dx dy \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi [\cos(\mu-m)x + \cos(\mu+m)x] [\cos(\nu-n)y + \cos(\nu+n)y] \, dx dy \\ &= \frac{1}{4} (\alpha_{\mu-m, \nu-n} + \alpha_{\mu-m, \nu+n} + \alpha_{\mu+m, \nu-n} + \alpha_{\mu+m, \nu+n}). \end{aligned}$$

Similarly,

$$\begin{aligned} B'_{\mu\nu} &= \frac{1}{4} (\beta_{\mu-m, \nu-n} + \beta_{\mu-m, \nu+n} + \beta_{\mu+m, \nu-n} + \beta_{\mu+m, \nu+n}), \\ C'_{\mu\nu} &= \frac{1}{4} (\gamma_{\mu-m, \nu-n} + \gamma_{\mu-m, \nu+n} + \gamma_{\mu+m, \nu-n} + \gamma_{\mu+m, \nu+n}), \\ D'_{\mu\nu} &= \frac{1}{4} (\delta_{\mu-m, \nu-n} + \delta_{\mu-m, \nu+n} + \delta_{\mu+m, \nu-n} + \delta_{\mu+m, \nu+n}). \end{aligned}$$

But ψ' satisfies all conditions under which (16) was derived; hence,

$$\mathfrak{A}_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \cdot \psi' \, dx dy = \sum_{\mu, \nu=0}^{\infty} (aA' + bB' + cC' + dD')_{\mu\nu} / \Lambda_{\mu\nu}.$$

If we apply the same line of reasoning to

$$\psi'' = \phi \sin mx \cos ny,$$

we find

$$\mathfrak{B}_{mn} = \sum_{\mu, \nu=0}^{\infty} (aA'' + bB'' + cC'' + dD'')_{\mu\nu} / \Lambda_{\mu\nu},$$

where

$$\begin{aligned} A''_{\mu\nu} &= \frac{1}{4} (-\beta_{\mu-m, \nu-n} - \beta_{\mu-m, \nu+n} + \beta_{\mu+m, \nu-n} + \beta_{\mu+m, \nu+n}), \\ B''_{\mu\nu} &= \frac{1}{4} (\alpha_{\mu-m, \nu-n} + \alpha_{\mu-m, \nu+n} - \alpha_{\mu+m, \nu-n} - \alpha_{\mu+m, \nu+n}), \\ C''_{\mu\nu} &= \frac{1}{4} (-\delta_{\mu-m, \nu-n} - \delta_{\mu-m, \nu+n} + \delta_{\mu+m, \nu-n} + \delta_{\mu+m, \nu+n}), \\ D''_{\mu\nu} &= \frac{1}{4} (\gamma_{\mu-m, \nu-n} + \gamma_{\mu-m, \nu+n} - \gamma_{\mu+m, \nu-n} - \gamma_{\mu+m, \nu+n}). \end{aligned}$$

Similarly,

$$\mathfrak{C}_{mn} = \sum_{\mu, \nu=0}^{\infty} (aA''' + bB''' + cC''' + dD''')_{\mu\nu} / \Lambda_{\mu\nu}$$

with

$$\begin{aligned} A'''_{\mu\nu} &= \frac{1}{4} (-\gamma_{\mu-m, \nu-n} + \gamma_{\mu-m, \nu+n} - \gamma_{\mu+m, \nu-n} + \gamma_{\mu+m, \nu+n}), \\ B'''_{\mu\nu} &= \frac{1}{4} (-\delta_{\mu-m, \nu-n} + \delta_{\mu-m, \nu+n} - \delta_{\mu+m, \nu-n} + \delta_{\mu+m, \nu+n}), \\ C'''_{\mu\nu} &= \frac{1}{4} (\alpha_{\mu-m, \nu-n} - \alpha_{\mu-m, \nu+n} + \alpha_{\mu+m, \nu-n} - \alpha_{\mu+m, \nu+n}), \\ D'''_{\mu\nu} &= \frac{1}{4} (\beta_{\mu-m, \nu-n} - \beta_{\mu-m, \nu+n} + \beta_{\mu+m, \nu-n} - \beta_{\mu+m, \nu+n}), \end{aligned}$$

and

$$\mathfrak{D}_{mn} = \sum_{\mu, \nu=0}^{\infty} (aA'''' + bB'''' + cC'''' + dD'')_{\mu\nu} / \Lambda_{\mu\nu}$$

with

$$\begin{aligned} A''''_{\mu\nu} &= \frac{1}{4}(\delta_{\mu-m, \nu-n} - \delta_{\mu-m, \nu+n} - \delta_{\mu+m, \nu-n} + \delta_{\mu+m, \nu+n}), \\ B''''_{\mu\nu} &= \frac{1}{4}(-\gamma_{\mu-m, \nu-n} + \gamma_{\mu-m, \nu+n} + \gamma_{\mu+m, \nu-n} - \gamma_{\mu+m, \nu+n}), \\ C''''_{\mu\nu} &= \frac{1}{4}(-\beta_{\mu-m, \nu-n} + \beta_{\mu-m, \nu+n} + \beta_{\mu+m, \nu-n} - \beta_{\mu+m, \nu+n}), \\ D''''_{\mu\nu} &= \frac{1}{4}(\alpha_{\mu-m, \nu-n} - \alpha_{\mu-m, \nu+n} - \alpha_{\mu+m, \nu-n} + \alpha_{\mu+m, \nu+n}). \end{aligned}$$

Before using these formulas Fourier's constants with negative subscripts must be transformed into such with positive ones by considering that

$$\begin{aligned} (\alpha, \beta, \gamma, \delta)_{-\kappa, \lambda} &= (\alpha, -\beta, \gamma, -\delta)_{\kappa\lambda}, \\ (\alpha, \beta, \gamma, \delta)_{-\kappa, -\lambda} &= (\alpha, -\beta, -\gamma, \delta)_{\kappa\lambda}, \\ (\alpha, \beta, \gamma, \delta)_{\kappa, -\lambda} &= (\alpha, \beta, -\gamma, -\delta)_{\kappa\lambda}. \end{aligned}$$

Equations Involving the Partial Derivatives of a Function of a Surface.*

BY CHARLES A. FISCHER.

Introduction.

In a former paper† I have defined the derivative of a function of a surface, and proved that if a function $L(S)$ has a derivative $L'(S; x, y)$, which is continuous and approached uniformly, the first variation of $L(S)$ can be given by the equation

$$\frac{dL(S)}{d\lambda} = \iint L'(S; x, y) \frac{dz(x, y)}{d\lambda} dx dy. \quad (1)$$

Conversely, it can be proved that if there is a function $L'(S; x, y)$, continuous in all its arguments, which satisfies this equation for every family of surfaces in a given neighborhood, then $L'(S; x, y)$ is the derivative of $L(S)$. In the present paper, functions depending not only on a surface, but also on the values taken by a function at every point of the surface, are considered. Such a function has two partial functional derivatives.

In the first section these derivatives are defined. In Section 2 the adjoint of a functional is discussed, and the conditions that functionals of a certain type be self-adjoint are found. In Section 3 are found the conditions that two given functionals be the partial derivatives of a function of a surface. The fourth section contains the condition of integrability for an equation involving these derivatives, and the equation is found which is satisfied by the function

$$\Phi = \iiint (f_x^2 + f_y^2 + f_z^2) dx dy dz,$$

where $f(x, y, z)$ is a potential function. In the last section the characteristics of such an equation are briefly discussed. Similar work for functions of lines has been done by Levy.‡

* Read before the American Mathematical Society, Feb. 26, 1916.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVI (1914), No. 3, p. 289.

‡ "Sur l'integration des equations aux derivees fonctionelle partielles," *Rendiconti del Circolo Matematico di Palermo*, Vol. 37 (1914), p. 113.

§ 1. *The Variation of a Function of a Surface.*

The equations of surfaces discussed in this paper will be given in parametric form; and, consequently, the definition of the derivative of a function of a surface must be modified slightly. The equations of the surface S will be

$$S: \quad x=x(u, v), \quad y=y(u, v), \quad z=z(u, v),$$

defined over a region Ω in the u, v -plane, and those of the varied surface,

$$S_\epsilon: \quad x=x(u, v)+Xn(u, v), \quad y=y(u, v)+Yn(u, v), \\ z=z(u, v)+Zn(u, v),$$

where X, Y and Z are the direction cosines of the normal to S . The function $n(u, v)$ is assumed to be of class $C^{(r)}$, to have a permanent sign, to vanish everywhere excepting in the region $(u_0-\epsilon < u < u_0+\epsilon; v_0-\epsilon < v < v_0+\epsilon)$, and the absolute values of its partial derivatives including those of order r are assumed to be less than ϵ . Then the derivative said to be approached with order r , will be defined as

$$L'(S; u, v) = \lim_{\epsilon \rightarrow 0} \frac{L(S_\epsilon) - L(S)}{\sigma},$$

where

$$\sigma = \int_{u_0-\epsilon}^{u_0+\epsilon} \int_{v_0-\epsilon}^{v_0+\epsilon} n(u, v) H(u, v) du dv, \quad (2)$$

$H du dv$ being the element of area. That is, $H = \sqrt{EG - F^2}$, where $E = \Sigma x_u^2$, $F = \Sigma x_u x_v$ and $G = \Sigma x_v^2$, the summation sign signifying that the expression is symmetrical in x, y and z . It will always be assumed that $H \neq 0$. If the function $\omega(x, y, z)$ in my paper already referred to,* is replaced by the new function $n(u, v, \lambda)$, equation (9) in that paper can be replaced by the equation,

$$\left. \frac{dL(S_\lambda)}{d\lambda} \right|_{\lambda=0} = \iint_\Omega L'(S; u, v) n_\lambda(u, v, 0) H du dv. \quad (3)$$

In order to make the variation normal to all of the surfaces S_λ , instead of to S_0 alone, these surfaces can be considered as defined by the partial differential equations

$$S_\lambda: \quad x_\lambda = Xn_\lambda, \quad y_\lambda = Yn_\lambda, \quad z_\lambda = Zn_\lambda, \quad (4)$$

with the initial conditions $x(u, v, 0) = x(u, v)$, $y(u, v, 0) = y(u, v)$, $z(u, v, 0) = z(u, v)$. The existence theorems for partial differential equations prove that $x(u, v, \lambda)$, $y(u, v, \lambda)$ and $z(u, v, \lambda)$ are determined uniquely, at least if

* Fischer, *loc. cit.*, p. 291.

$n(u, v, \lambda)$, $x(u, v)$, $y(u, v)$ and $z(u, v)$ are analytic, and cases where they are not so determined will not be considered here. This change in the equations of S_λ does not affect equation (3).

A function $\Phi(n, f)$ will now be considered which depends on the surface designated by the argument n , and on all of the values taken by another function $f(u, v)$ at points of the surface. Such a function may have two partial functional derivatives,

$$\Phi'_n(0, f; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(0, f)}{\sigma}, \quad (5)$$

and

$$\Phi'_f(n, f_0; u, v) = \lim_{\epsilon=0} \frac{\Phi(n, f) - \Phi(n, f_0)}{\sigma'}, \quad (6)$$

where σ is defined by equation (2), and

$$\sigma' = \int_{u_0-\epsilon}^{u_0+\epsilon} \int_{v_0-\epsilon}^{v_0+\epsilon} (f(u, v) - f_0(u, v)) H du dv.$$

It is assumed, of course, that $f - f_0$ has the properties already assumed for n . If these partial derivatives are continuous in all arguments and approached uniformly, the equation

$$\frac{d\Phi(n, f)}{d\lambda} = \iint_{\Omega} [\Phi'_n(n, f; u, v) n_\lambda + \Phi'_f(n, f; u, v) f_\lambda] H du dv, \quad (7)$$

which is analogous to equation (3), must be satisfied, if $n(u, v, \lambda)$ and $f(u, v, \lambda)$ and all of their partial derivatives considered are continuous everywhere, and independent of λ along the boundary of Ω , and if $n_\lambda(u, v, \lambda)$ and $f_\lambda(u, v, \lambda)$ are approached uniformly. It was also assumed to simplify the proof that the function which corresponds to $n(u, v, \lambda') - n(u, v, \lambda)$ should have a permanent sign,* but this is not essential.

§ 2. The Adjoint of a Functional.

The adjoint of a functional of a line has been defined by Levy,† and it was found useful in deriving the condition of integrability for equations involving functional derivatives. The adjoint of a functional of a surface will be defined similarly.

If there are two functionals $L(f)$ and $\bar{L}(f)$ such that the equation

$$\iint_{\Omega} g(u, v) L(f) H du dv = \iint_{\Omega} f(u, v) \bar{L}(g) H du dv \quad (8)$$

* Fischer, *loc. cit.*, p. 291.

† Levy, *loc. cit.*, p. 115.

is satisfied for every pair of functions f and g of class $C^{(n)}$ which vanish together with their partial derivatives along the boundary of Ω , then \bar{L} is said to be the adjoint of L , and vice versa. The proof that a functional can not have two distinct adjoints is essentially the same as for functionals of lines,* and will not be repeated. If a functional has an adjoint it must be linear, since if \bar{L} is the adjoint of L the equations

$$\begin{aligned}\iint_{\Omega} g L(af_1 + bf_2) H du dv &= a \iint_{\Omega} f_1 \bar{L}(g) H du dv + b \iint_{\Omega} f_2 \bar{L}(g) H du dv \\ &= \iint_{\Omega} g [aL(f_1) + bL(f_2)] H du dv,\end{aligned}$$

must be satisfied for every function $g(u, v)$. It follows that

$$L(af_1 + bf_2) = aL(f_1) + bL(f_2),$$

which is the condition that L be linear. Every linear functional can be expressed in the form

$$\lim_{\epsilon=0} \iint_{\Omega} F(u, v, u_1, v_1, \epsilon) f(u_1, v_1) H(u_1, v_1) du_1 dv_1. \dagger$$

Its adjoint will be

$$\lim_{\epsilon=0} \iint_{\Omega} F(u_1, v_1, u, v, \epsilon) f(u_1, v_1) H(u_1, v_1) du_1 dv_1,$$

provided this limit exists and is approached uniformly. However, there are linear functionals which have no adjoints.

A large class of functionals are expressible in the form

$$\begin{aligned}L(f) &= \iint_{\Omega} F(u, v, u_1, v_1) f(u_1, v_1) H(u_1, v_1) du_1 dv_1 \\ &\quad + \sum_{i,j=0}^{i+j=m} A_{ij}(u, v) \frac{\partial^{i+j} f(u, v)}{\partial u^i \partial v^j}.\end{aligned}\quad (9)$$

The adjoint is

$$\begin{aligned}\bar{L}(f) &= \iint_{\Omega} F(u_1, v_1, u, v) f(u_1, v_1) H(u_1, v_1) du_1 dv_1 \\ &\quad + \frac{1}{H} \sum_{i,j=0}^{i+j=m} (-1)^{i+j} \frac{\partial^{i+j} A_{ij} f H}{\partial u^i \partial v^j}.\end{aligned}\quad (10)$$

This can be verified by substituting in equation (8), and applying Green's theorem repeatedly. If the function (9) is self-adjoint, the equation

$$F(u, v, u_1, v_1) = F(u_1, v_1, u, v)$$

will be satisfied, and also the equations obtained by equating the coefficients of $\partial^{k+l}/\partial u^k \partial v^l$ in the equation

$$\sum_{i,j=0}^{i+j=m} A_{ij} \frac{\partial^{i+j} f}{\partial u^i \partial v^j} = \frac{1}{H} \sum_{i,j=0}^{i+j=m} (-1)^{i+j} \frac{\partial^{i+j} A_{ij} H f}{\partial u^i \partial v^j}.$$

* Levy, *loc. cit.*, p. 116.

† Hadamard, "Leçons sur le calcul des variations," p. 303.

If $m=2$ these equations are seen to be

$$\left. \begin{aligned} A_{20} &= A_{20}, \quad A_{11} = A_{11}, \quad A_{02} = A_{02}, \\ A_{10} &= -A_{10} + \frac{1}{H} \left(2 \frac{\partial A_{20} H}{\partial u} + \frac{\partial A_{11} H}{\partial v} \right), \\ A_{01} &= -A_{01} + \frac{1}{H} \left(\frac{\partial A_{11} H}{\partial u} + 2 \frac{\partial A_{02} H}{\partial v} \right), \\ A_{00} &= A_{00} + \frac{1}{H} \left(\frac{\partial^2 A_{20} H}{\partial u^2} + \frac{\partial^2 A_{11} H}{\partial u \partial v} + \frac{\partial^2 A_{02} H}{\partial v^2} - \frac{\partial A_{10} H}{\partial u} - \frac{\partial A_{01} H}{\partial v} \right). \end{aligned} \right\} \quad (11)$$

If the values of A_{10} and A_{01} are substituted in the last of these equations it becomes an identity. It follows that if A_{20} , A_{11} , A_{02} and A_{00} are taken arbitrarily, A_{10} and A_{01} can be determined so that the functional will be self-adjoint. It is evident from the equations similar to (11) for higher values of m , that m must be even, and that the functions A_{ij} with $i+j$ odd are determined by the others. It will be proved a little later that the functions A_{ij} with $i+j$ even can always be taken arbitrarily. The following theorems are easily verified.*

If $L(f)$ and $\bar{L}(f)$ are adjoint, then $h(u, v)L(k(uv)f)$ is adjoint to $k(u, v)\bar{L}(h(u, v)f)$, where $h(u, v)$ and $k(u, v)$ are arbitrary. It follows that if $L(f)$ is self-adjoint, $h(u, v)L(h(u, v)f)$ is also.

If $L(f)$ and $M(f)$ have the adjoints $\bar{L}(f)$ and $\bar{M}(f)$, respectively, then $L(M(f))$ is the adjoint of $\bar{M}(\bar{L}(f))$. Similarly, if $L(f)$ is self-adjoint, $\bar{M}(L(M(f)))$ is also.

From the last statement if $L(f) = \phi(u, v)f(u, v)$, where $\phi(u, v)$ is arbitrary, and $M(f) = \partial^{i+j}f(u, v)/\partial u^i \partial v^j$, then the functional

$$\bar{M}(L(M(f))) = \frac{(-1)^{i+j}}{H} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left(H \phi \frac{\partial^{i+j} f}{\partial u^i \partial v^j} \right) \quad (12)$$

must be self-adjoint. This may be expressed as

$$\sum_{k=i}^{2i} \sum_{l=j}^{2j} A_{kl} \frac{\partial^{k+l} f}{\partial u^k \partial v^l},$$

where $A_{2i2j} = (-1)^{i+j} \phi$. The function

$$\phi \frac{\partial^2 f}{\partial u \partial v} + \frac{1}{2H} \left(\frac{\partial \phi H}{\partial v} \cdot \frac{\partial f}{\partial u} + \frac{\partial \phi H}{\partial u} \cdot \frac{\partial f}{\partial v} \right)$$

* Compare with Levy, *loc. cit.*, p. 116.

is self-adjoint and can be used instead of $\phi(u, v)f$. It follows that

$$\bar{M}(L(M(f))) = \frac{(-1)^{i+j}}{H} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left(H\phi \frac{\partial^{i+j+2}f}{\partial u^{i+1} \partial v^{j+1}} + \frac{1}{2} \frac{\partial \phi H}{\partial v} \cdot \frac{\partial^{i+j+1}f}{\partial u^{i+1} \partial v^j} + \frac{1}{2} \frac{\partial \phi H}{\partial u} \cdot \frac{\partial^{i+j+1}f}{\partial u^i \partial v^{j+1}} \right) \quad (13)$$

is also self-adjoint. This is equal to

$$\sum_{k=i}^{2i+1} \sum_{l=j}^{2j+1} A_{kl} \frac{\partial^{k+l}f}{\partial u^k \partial v^l},$$

where $A_{2i+1, 2j+1} = (-1)^{i+j}\phi$, which is arbitrary. A self-adjoint functional of the type

$$\sum_{i, j=0}^{i+j=m} A_{ij} \frac{\partial^{i+j}f}{\partial u^i \partial v^j} \quad (14)$$

can then be formed by adding those of types (12) and (13), in which the functions A_{ij} with $i+j$ even are arbitrary, as has been stated previously. Since the functions A_{ij} with $i+j$ odd are determined uniquely by the condition that the functional be self-adjoint, every self-adjoint functional of type (14) is expressible as the sum of a finite number of functionals of types (12) and (13).

§ 3. *The Conditions that Given Functionals Be the Derivatives of Functions of Surfaces.*

In finding the variation of a functional depending on the surface S_λ defined by equations (4), the function $H(u, v)$ must be considered a function of λ also. Its derivative will now be calculated. By definition,

$$\frac{\partial H}{\partial \lambda} = \frac{1}{2H} \left(G \frac{\partial E}{\partial \lambda} + E \frac{\partial G}{\partial \lambda} - 2F \frac{\partial F}{\partial \lambda} \right), \quad (15)$$

and

$$\frac{\partial E}{\partial \lambda} = 2 \sum x_u x_{u\lambda} = 2 \sum (x_u X_u n_\lambda + x_u X n_{u\lambda}) = -2D n_\lambda,$$

since

$$\sum x_u X = 0, \text{ and } \sum x_u X_u = -D.*$$

If $\partial F/\partial \lambda$ and $\partial G/\partial \lambda$ are evaluated in the same way, equation (15) becomes

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{H} (GD + ED'' - 2FD') n_\lambda = -K_m H n_\lambda,$$

where K_m is the mean curvature of the surface.†

* The functions D , D' and D'' are called the fundamental coefficients of the second order. See Eisenhart, "Differential Geometry," p. 115.

† Eisenhart, *loc. cit.*, p. 123.

The condition that a given functional $\Psi_1(S; u, v)$ be the derivative of some function $\Psi(S)$ will now be derived by a method given by Volterra.* It will be assumed that Ψ_1 has a differential. That is to say that there is a linear functional $L(n_\lambda)$ such that $\partial\Psi_1/\partial\lambda = L(n_\lambda)$. A two parameter family of surfaces $S_{\lambda, \lambda'}$ will be determined by a function $n(u, v, \lambda, \lambda')$ having the properties assumed for $n(u, v, \lambda)$. Then a closed curve,

$$l: \quad \lambda = \lambda(s), \quad \lambda' = \lambda'(s),$$

will be chosen bounding a small region ω in the λ, λ' -plane. It follows from Green's theorem that

$$\begin{aligned} \int_1 ds \iint_{\omega} \Psi_1(S; u, v) H \left(n_\lambda \frac{d\lambda}{ds} + n_{\lambda'} \frac{d\lambda'}{ds} \right) dudv \\ = \iint_{\omega} d\lambda d\lambda' \iint_{\omega} \left\{ \frac{\partial}{\partial\lambda} (\Psi_1 H n_{\lambda'}) - \frac{\partial}{\partial\lambda'} (\Psi_1 H n_\lambda) \right\} dudv. \end{aligned}$$

This is equivalent to the equation

$$\int_1 ds \iint_{\omega} \Psi_1(S; u, v) \frac{dn}{ds} H dudv = \iint_{\omega} d\lambda d\lambda' \iint_{\omega} (L(n_\lambda) n_{\lambda'} - L(n_{\lambda'}) n_\lambda) H dudv. \quad (16)$$

If $\Psi_1(S; u, v)$ is the derivative of some function $\Psi(S)$, the left member of this equation is equal to

$$\int_1 \frac{d\Psi}{ds} ds,$$

which vanishes for every choice of n_λ and $n_{\lambda'}$. Consequently, the right member must vanish, and $L(n_\lambda)$ is self-adjoint. Conversely, if it is self-adjoint the right member will vanish. The value of Ψ can then be taken arbitrarily for one surface S_0 , and for S_λ

$$\Psi(S_\lambda) = \Psi(S_0) + \int_0^\lambda d\lambda \iint_{\omega} \Psi_1(S_\lambda; u, v) n_\lambda H dudv.$$

There are analogous conditions which must be satisfied if two given functionals $\Phi_n(n, f; u, v)$ and $\Phi_f(n, f; u, v)$ are to be equal to the partial derivatives of some function $\Phi(n, f)$. These functionals will be assumed to have differentials which satisfy the equations

$$\frac{\partial\Phi_n}{\partial\lambda} = \Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda), \quad \frac{\partial\Phi_f}{\partial\lambda} = \Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda). \quad (17)$$

* Volterra, "Fonctions de lignes," p. 45.

Equation (16) must then be replaced by the analogous equation

$$\begin{aligned} \int_1 ds \int \int \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds} \right) H du dv \\ = \int \int \omega d\lambda d\lambda' \int \int \left\{ (\Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda)) n_{\lambda'} + (\Phi_{fn}(n_\lambda) + \Phi_{ff}(f_\lambda)) f_{\lambda'} \right. \\ - (\Phi_{nn}(n_{\lambda'}) + \Phi_{nf}(f_{\lambda'})) n_\lambda - (\Phi_{fn}(n_{\lambda'}) + \Phi_{ff}(f_{\lambda'})) f_\lambda + \Phi_f(f_{\lambda\lambda} - f_{\lambda\lambda'}) \\ \left. - K_m \Phi_f(f_{\lambda'} n_\lambda - f_\lambda n_{\lambda'}) \right\} H du dv. \end{aligned} \quad (18)$$

The derivatives $f_{\lambda\lambda}$ and $f_{\lambda\lambda'}$ will not in general be equal. Their difference will be

$$f_{\lambda\lambda} - f_{\lambda\lambda'} = \Sigma (f_u u_x + f_v v_x + f_n n_x) (x_{\lambda\lambda} - x_{\lambda\lambda'}). \quad (19)$$

If the equations

$$u_x x_u + u_y y_u + u_z z_u = 1, \quad u_x x_v + u_y y_v + u_z z_v = 0, \quad u_x X + u_y Y + u_z Z = 0,$$

are solved for u_x , u_y and u_z , the equations

$$\Sigma u_x^2 = \frac{1}{H^2} \Sigma [(1 - X^2) x_v^2 - 2YZ y_v z_v] = \frac{1}{H^2} [G - (\Sigma X x_v)^2] = \frac{G}{H^2} \quad (20)$$

are easily derived. Similarly, it can be proved that

$$\Sigma u_x v_x = \frac{-F}{H^2}, \quad \Sigma v_x^2 = \frac{E}{H^2}, \quad n_x = X, \quad \Sigma u_x n_x = \Sigma v_x n_x = 0. \quad (21)$$

To evaluate $x_{\lambda\lambda} - x_{\lambda\lambda'}$, equations (4) must be differentiated with respect to λ' , and subtracted from the same equations with λ and λ' interchanged. This gives the equations

$$\begin{aligned} x_{\lambda\lambda} - x_{\lambda\lambda'} &= \frac{1}{H} \left| \frac{Y y_v}{Z z_v} \right| \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) + \frac{1}{H} \left| \frac{y_u Y}{z_u Z} \right| \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right) \\ &= -u_x \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) - v_x \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right). \end{aligned}$$

If these values are substituted in equation (19) it can be reduced to the form

$$f_{\lambda\lambda} - f_{\lambda\lambda'} = \frac{1}{H^2} \left\{ (-f_u G + f_v F) \left(\frac{\partial n_\lambda}{\partial u} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial u} n_\lambda \right) + (f_u F - f_v E) \left(\frac{\partial n_\lambda}{\partial v} n_{\lambda'} - \frac{\partial n_{\lambda'}}{\partial v} n_\lambda \right) \right\}.$$

If this expression is substituted in equation (18) it becomes

$$\begin{aligned} \int_1 ds \int \int \left(\Phi_n \frac{dn}{ds} + \Phi_f \frac{df}{ds} \right) H du dv &= \int \int \omega d\lambda d\lambda' \int \int \left\{ \left[\Phi_{nn}(n_\lambda) \right. \right. \\ &+ \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_\lambda}{\partial u} + (f_u F - f_v E) \frac{\partial n_\lambda}{\partial v} \right) \Big] n_{\lambda'} \\ &- \left[\Phi_{nn}(n_{\lambda'}) + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial n_{\lambda'}}{\partial u} + (f_u F - f_v E) \frac{\partial n_{\lambda'}}{\partial v} \right) \right] n_\lambda \\ &+ [\Phi_{nf}(f_\lambda) + K_m \Phi_f f_\lambda] n_{\lambda'} - \Phi_{fn}(n_{\lambda'}) f_\lambda + \Phi_{fn}(n_\lambda) f_{\lambda'} \\ &\left. - [\Phi_{nf}(f_{\lambda'}) + K_m \Phi_f f_{\lambda'}] n_\lambda + \Phi_{ff}(f_\lambda) f_{\lambda'} - \Phi_{ff}(f_{\lambda'}) f_\lambda \right\} H du dv. \end{aligned}$$

If there is a function $\Phi(n, f)$ which has the partial derivatives $\Phi'_n = \Phi_n$ and $\Phi'_f = \Phi_f$, the left member of the last equation, and consequently the right member, must vanish for every choice of the functions n_λ , n_ν , f_λ and f_ν . The necessary and sufficient conditions for this are that the functionals

$$\Phi_{nn}(g) + \frac{\Phi_f}{H^2} \left((-f_u G + f_v F) \frac{\partial g}{\partial u} + (f_u F - f_v E) \frac{\partial g}{\partial v} \right)$$

and $\Phi_{ff}(g)$ be self-adjoint, and that $\Phi_{nf}(g) + K_m \Phi_f g$ be the adjoint of $\Phi_{fn}(g)$. The similar conditions for functions of lines are given by Levy.*

§ 4. Equations Involving Partial Functional Derivatives.

The condition of integrability of an equation such as

$$\Phi'_n(n, f; u, v) = W(n, f, \Phi'_f; \Phi, u, v) \quad (22)$$

will now be found.† If there is an integral $\Phi(n, f)$ which is equal to an arbitrarily given function $\Psi(f)$ when $n=0$, the equation is said to be completely integrable. It will be assumed that the derivative $\Psi'(f; u, v)$ is continuous and approached uniformly, and that Ψ' and W have differentials, implying the equations

$$\frac{d\Psi'}{d\lambda} = \Psi_{ff}(f_\lambda), \quad \frac{dW}{d\lambda} = Q\left(\frac{d\Phi'_f}{d\lambda}\right) + M(n_\lambda) + L(f_\lambda), \quad (23)$$

for every choice of the functions $n(u, v, \lambda)$ and $f(u, v, \lambda)$ such that f_λ , n_λ and $\partial\Phi'_f/\partial\lambda$ are of class $C^{(r)}$ in u and v .

If equation (22) is differentiated with respect to λ , and values substituted from equations (17) and (23), it becomes

$$\Phi_{nn}(n_\lambda) + \Phi_{nf}(f_\lambda) = Q(\Phi_{fn}(n_\lambda)) + Q(\Phi_{ff}(f_\lambda)) + M(n_\lambda) + L(f_\lambda).$$

It follows that for any function $g(u, v)$,

$$\Phi_{nn}(g) = Q(\Phi_{fn}(g)) + M(g),$$

and

$$\Phi_{nf}(g) = Q(\Phi_{ff}(g)) + L(g).$$

On the surface defined by $n=0$, the functional Φ_{ff} is equal to Ψ_{ff} which must be self-adjoint. The functional $\Phi_{fn}(g)$ is the adjoint of $\Phi_{nf}(g) + K_m \Phi_f g$, and, consequently,

$$\Phi_{fn}(g) = \Phi_{ff}(\bar{Q}(g)) + \bar{L}(g) + K_m \Phi'_f g. \quad (24)$$

* Levy, *loc. cit.*, p. 120.

† Compare with Levy, *loc. cit.*, pp. 122-123.

The other functional which was proved to be self-adjoint in the last section is now equal to

$$Q(\Phi_H(\bar{Q}(g))) + Q(\bar{L}(g)) + Q(K_m \Phi'_i g) + M(g) + \frac{\Phi'_i}{H^2} \left((-f_u G + f_v F) \frac{\partial g}{\partial u} + (f_u F - f_v E) \frac{\partial g}{\partial v} \right).$$

The first term is self-adjoint as was proved in § 2. A necessary condition that equation (22) be completely integrable is, therefore, that the functional

$$Q(\bar{L}(g)) + Q(K_m \Phi'_i g) + M(g) + \frac{\Phi'_i}{H^2} \left((-f_u G + f_v F) \frac{\partial g}{\partial u} + (f_u F - f_v E) \frac{\partial g}{\partial v} \right) \quad (25)$$

be self-adjoint.

To illustrate the theory just developed, the equation will be found which must be satisfied by the function

$$\Phi = \iiint_R (f_x^2 + f_y^2 + f_z^2) dx dy dz, \quad (26)$$

where $f(x, y, z)$ is a solution of the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \quad (27)$$

The value of f inside the region R is then determined by the values it takes on the surface S which bounds R . If S is considered as fixed, while the boundary values of $f(x, y, z)$ are varied, the equation

$$\frac{d\Phi}{d\lambda} = 2 \iiint_R \Sigma f_x f_{x\lambda} dx dy dz$$

will be satisfied. This can be reduced by means of Green's formula and equation (27) to the form

$$\frac{d\Phi}{d\lambda} = 2 \iint_S f_n f_{\lambda} H du dv.$$

It follows that

$$\Phi'_i(n, f; u, v) = 2f_n(x(u, v), y(u, v), z(u, v)). \quad (28)$$

If the value of $f(x, y, z)$ is fixed at each point of R , while S is varied, the boundary values of $f(x, y, z)$ will vary according to the law $f_{\lambda} = f_n n_{\lambda}$. In this case it follows from equations (7) and (26) that

$$\iint (\Phi'_n + f_n \Phi'_i) n_{\lambda} H du dv = \iint (f_x^2 + f_y^2 + f_z^2) n_{\lambda} H du dv,$$

and, consequently,

$$\Phi'_n + f_n \Phi'_i = \Sigma f_x^2.$$

If f_x is replaced by $f_u u_x + f_v v_x + f_n n_x$ and equations (20) and (21) applied, the right member of this equation becomes

$$\frac{1}{H^2} (f_u^2 G - 2f_u f_v F + f_v^2 E) + f_n^2.$$

Substituting the value of f_n from equation (28),

$$\Phi'_n = -\frac{1}{2}\Phi'^2 + \frac{1}{H^2} (f_u^2 G - 2f_u f_v F + f_v^2 E).$$

This is the desired equation.

It can easily be proved that the condition of integrability is satisfied. The functionals Q , M and L are, evidently,

$$\begin{aligned} Q(g) &= -\frac{1}{2}\Phi'_n g, \quad M(g) = 0, \\ L(g) &= \frac{2}{H^2} \left[(f_u G - f_v F) \frac{\partial g}{\partial u} + (-f_u F + f_v E) \frac{\partial g}{\partial v} \right]. \end{aligned}$$

If these values are substituted in the expression (25) the terms involving $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ cancel and the other terms are self-adjoint.

§ 5. Characteristics.

There is a set of functionals which have the same relation to a solution of equation (22) that the characteristics have to a solution of a partial differential equation. The characteristics of equations involving partial derivatives of functions of lines have been discussed by Levy.* There are characteristics of various orders, but the most interesting ones are the "caractéristiques de première espèce," and these are the only ones which will be considered here. After they are defined it will be proved that if an integral contains an element of a characteristic, it contains the whole characteristic. An element and an integral will be defined first.

An element is any set of functions $n(u, v)$, $f(u, v)$, $\Phi(n, f)$, $\Phi'_n(n, f; u, v)$ and $\Phi'_f(n, f; u, v)$ which satisfy equation (22).

An integral is a function $\Phi(n, f)$ whose partial derivatives satisfy equation (22) for every pair of admissible functions $n(u, v)$ and $f(u, v)$.

A characteristic is a set of functions $g(u, v)$, $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$ which satisfy the equations

$$\begin{aligned} \frac{dg}{d\lambda} &= -\bar{Q}(n_\lambda), \\ \frac{d\Psi}{d\lambda} &= \iint_a (\Psi'_n - Q(\Psi'_f)) n_\lambda H du dv, \end{aligned} \tag{29}$$

$$\frac{d\Psi'_n}{d\lambda} = Q(\bar{L}(n_\lambda)) + Q(K_m \Psi'_f n_\lambda) - L(\bar{Q}(n_\lambda)) + M(n_\lambda), \tag{30}$$

$$\frac{d\Psi'_f}{d\lambda} = \bar{L}(n_\lambda) + K_m \Psi'_f n_\lambda, \tag{31}$$

* Levy, *loc. cit.*, Ch. II.

where the function Ψ is used instead of Φ in deriving Q , L and M . If $\Phi(n, f)$ is an integral the equations

$$\frac{d\Phi}{d\lambda} = \iint_0 (\Phi'_n n_\lambda + \Phi'_f f_\lambda) H du dv, \quad (32)$$

$$\frac{d\Phi'_n}{d\lambda} = Q \left(\frac{d\Phi'_f}{d\lambda} \right) + L(f_\lambda) + M(n_\lambda), \quad (33)$$

and

$$\frac{d\Phi'_f}{d\lambda} = \Phi_{ff}(f_\lambda) + \Phi_{fn}(\bar{Q}(n_\lambda)) + \bar{L}(n_\lambda) + K_m \Phi'_f n_\lambda, \quad (34)$$

will be satisfied, on account of equations (7), (17), (22), (23) and (24). If the value given for g_λ is substituted for f_λ in equations (32), (33) and (34), they become equivalent to (29), (30) and (31). It follows that if the integral $\Phi(n, g)$ and its derivatives are equal to $\Psi(n)$, $\Psi'_n(n; u, v)$ and $\Psi'_f(n; u, v)$, respectively, for $\lambda = \lambda_0$, they are equal for all values of λ for which the equations of the characteristics have unique solutions.

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Invariants and Covariants of the Cremona Cubic Surface.

By C. P. SOUSLEY.

Introduction.

If a, b, \dots, f are properly chosen cubic curves on P^2_3 , i. e., on six points in a plane, the cubic surface mapped on the plane by these curves may be given by the equation

$$a^3 + b^3 + \dots + f^3 = 0,$$

where the variables are subject to the relations

$$\begin{aligned} a + b + \dots + f &= 0, \\ \bar{a}a + \bar{b}b + \dots + \bar{f}f &= 0. \end{aligned}$$

This form is known as the hexahedral cubic surface of Cremona.*

As to the invariants and covariants of the Sylvester pentahedral form of the surface much is known and given in explicit form in Salmon's geometry of three dimensions. Nothing, however, is known as to the invariants and covariants of this Cremona form.

The purpose of this paper is to obtain some of these invariants and covariants and to outline the further steps necessary for the determination of the invariants and linear covariants.

The results are important in determining the lines on a general cubic surface. In particular for the Cremona form, the equations of the lines are known. If, then, one can find a typical representation for the general cubic surface in terms of the Cremona form and can determine the irrational invariants for the Cremona form, then the required lines of the general surface can be determined from the known lines of the Cremona cubic surface.

This requires the calculation of the invariants and linear covariants for the Cremona form and their identification with the corresponding invariants and covariants of the general cubic surface.

The results obtained are valuable also for any study that involves the behavior of the lines of a cubic surface with reference to the covariants of the surface.

Section 1.

Given in S_3 the cubic surface $(\alpha x)^3 = (\beta x)^3 = \dots$ a known comitant $|\alpha\beta\gamma\xi| \dots (\delta x) \dots$ becomes $|\alpha\beta\gamma\eta\zeta\xi| \dots (\delta x)$, when the surface is taken in S_5 , and the variables are subject to the conditions $\sum_1^6 \eta_1 x_1 = 0$, $\sum_1^6 \zeta_1 x_1 = 0$.

* *Mathematische Annalen*, Vol. XIII.

This process is known as the transference principle of Clebsch. The straightforward method for obtaining the corresponding comitant in S_3 would be to eliminate two variables, thus getting the surface in S_3 , and for this form to calculate the comitant, but this would give an unsymmetrical result; whereas, by the Clebsch transference principle we obtain a symmetrical result.

The process of contravariant differentiation is not affected by going up to S_5 from S_3 in this way. This fact is noted immediately on performing corresponding operations in both dimensions.

Let us examine the degree of the invariants and covariants of the Cremona form. A covariant of a cubic surface of degree d , weight w , class c , order σ , has $3d$ symbols α ; distributed in $w-c$ symbols $|\alpha\beta\gamma\delta|$, c symbols $|\alpha\beta\gamma\xi|$ and σ symbols (αx) , whence $3d = 4(w-c) + 3c + \sigma$, or $3d = 4w - c + \sigma$. These symbols become by Clebsch's principle of transference $|\alpha\beta\gamma\delta\eta\zeta|$, $|\alpha\beta\gamma\xi\eta\zeta|$, (αx) , where for the Cremona cubic surface the coefficients α are now numerical and the coefficients η all are 1 and the coefficients ζ are $\bar{a}, \bar{b}, \dots, \bar{f}$. Hence, if d' is the degree in $\bar{a}, \bar{b}, \dots, \bar{f}$ of the covariant for the Cremona form, we have

$$d' = w = \frac{1}{4}(3d - \sigma + c). \quad (1)$$

Hence for this canonical form, the invariants have the degree $d' = 6, 12, 18, 24, 30, 75$; the linear covariants have the degree $d' = 8, 14, 20, 32$.

The invariants of the surface being invariants of P_6^2 , must be expressible in terms of the rational and symmetric functions of the irrational system \bar{a}, \dots, \bar{f} of P_6^2 . The linear covariants of the surface can be expressed in terms of the functions

$$K_i = \bar{a}^i a + \bar{b}^i b + \dots + \bar{f}^i f, \quad (i=2, \dots, 5). \quad (2)$$

Let us now calculate for P_6^2 , referred to a special coordinate system, the irrational system \bar{a}, \dots, \bar{f} , and from these form the rational system a_2, \dots, a_6 ,* which are the elementary symmetric functions of \bar{a}, \dots, \bar{f} .

Section 2.

If we take P_6^2 in the prepared form

$$\begin{array}{ll} (1) & 1 \ 0 \ 0, \\ (2) & 0 \ 1 \ 0, \\ (3) & 0 \ 0 \ 1, \end{array} \quad \begin{array}{ll} (4) & 1 \ 1 \ 1, \\ (5) & x \ y \ u, \\ (6) & z \ t \ u, \end{array} \quad (3)$$

we have (*loc. cit.*, p. 170)

$$\left. \begin{array}{ll} 3\bar{a} = \rho - 3(ux + ut), & 3\bar{d} = \rho - 3(uy + uz), \\ 3\bar{b} = \rho - 3(ux + yz), & 3\bar{e} = \rho - 3(uy + xt), \\ 3\bar{c} = \rho - 3(ut + yz), & 3\bar{f} = \rho - 3(uz + xt), \end{array} \right\} \quad (4)$$

$$\rho = u(x + y + z + t) + xt + yz.$$

* Coble, "Point Sets and Allied Cremona Groups," *Trans. American Math. Soc.*, Vol. XVI (1915), p. 155; in particular § 5.

Let us introduce the following notation:

$$\left. \begin{aligned} \gamma_1 &= x+y+z+t, \\ \gamma_2 &= (x+t)(y+z), \quad \bar{\gamma}_2 = yz+xt, \\ \gamma_3 &= xyz+yzt+xyt+xzt, \\ \gamma_4 &= x y z t. \end{aligned} \right\} \quad (5)$$

For purposes of calculation it is more convenient to use the sums of the powers of \bar{a}, \dots, \bar{f} for the rational invariants of P_0^2 , rather than their symmetric functions, i. e., if $p_2 = \Sigma \bar{a}^2$, $p_3 = \Sigma \bar{a}^3$, \dots , $p_5 = \Sigma \bar{a}^5$, and if we substitute the values of \bar{a}, \dots, \bar{f} , given by (4), we find

$$\begin{aligned} 9p_2 &= 6[(2\gamma_1^2 - 3\bar{\gamma}_2 - 6\gamma_2)u^2 - (2\gamma_1\bar{\gamma}_2 - 3\gamma_3)u + 2(\bar{\gamma}_2^2 - 3\gamma_4)], \\ 27p_3 &= 3[(9\gamma_1\bar{\gamma}_2 + 18\gamma_1\gamma_2 - 4\gamma_1^3 - 27\gamma_3)u^3 + 3(2\gamma_1^2\bar{\gamma}_2 - 6\bar{\gamma}_2^2 \\ &\quad - 3\gamma_1\gamma_3 - 3\bar{\gamma}_2\gamma_2 + 36\gamma_4)u^2 + 3(2\gamma_1\bar{\gamma}_2^2 - 3\gamma_1\gamma_4 - 3\bar{\gamma}_2\gamma_3)u \\ &\quad + 2\bar{\gamma}_2(9\gamma_4 - 2\bar{\gamma}_2^2)], \\ 81p_4 &= 18[(2\gamma_1^4 - 12\gamma_1^2\gamma_2 - 6\gamma_1^2\bar{\gamma}_2 + 18\gamma_2^2 + 9\bar{\gamma}_2^2 + 18\bar{\gamma}_2\gamma_2 - 18\gamma_4)u^4 \\ &\quad + 2(3\gamma_1\bar{\gamma}_2^2 - 2\gamma_1^3\bar{\gamma}_2 + 6\gamma_1\bar{\gamma}_2\gamma_2 - 9\bar{\gamma}_2\gamma_3 - 9\gamma_2\gamma_3 + 3\gamma_1^2\gamma_3 + 9\gamma_1\gamma_4)u^3 \\ &\quad + 3(2\gamma_1^2\bar{\gamma}_2^2 - 2\bar{\gamma}_2^3 - 6\gamma_1\bar{\gamma}_2\gamma_3 + 6\bar{\gamma}_2\gamma_4 + 7\gamma_3^2 - 10\gamma_2\gamma_4)u^2 \\ &\quad + 2(3\bar{\gamma}_2^2\gamma_3 - 2\gamma_1\bar{\gamma}_2^3 + 6\gamma_1\bar{\gamma}_2\gamma_4 - 9\gamma_3\gamma_4)u + 2(\bar{\gamma}_2^4 - 6\bar{\gamma}_2^2\gamma_4 + 9\gamma_4^2)], \\ 243p_5 &= 5[3(-4\gamma_1^5 + 30\gamma_1^3\gamma_2 + 15\gamma_1^3\bar{\gamma}_2 - 27\gamma_1^2\gamma_3 - 54\gamma_1\gamma_2^2 \\ &\quad - 27\gamma_1\bar{\gamma}_2^2 - 54\gamma_1\bar{\gamma}_2\gamma_2 - 27\gamma_1\gamma_4 + 81\bar{\gamma}_2\gamma_3 + 81\gamma_2\gamma_3)u^5 \\ &\quad + 3(10\gamma_1^4\bar{\gamma}_2 - 45\gamma_1^2\bar{\gamma}_2\gamma_2 - 36\gamma_1^2\bar{\gamma}_2^2 + 54\bar{\gamma}_2^3 - 270\bar{\gamma}_2\gamma_4 - 15\gamma_1^3\gamma_3 \\ &\quad + 108\gamma_1^2\gamma_4 + 108\gamma_1\bar{\gamma}_2\gamma_3 + 54\gamma_1\gamma_2\gamma_3 + 27\bar{\gamma}_2\gamma_2^2 + 216\gamma_2\gamma_4 - 108\gamma_3^2)u^4 \\ &\quad + 6(9\gamma_1^2\bar{\gamma}_2\gamma_3 - 27\bar{\gamma}_2^2\gamma_3 + 9\gamma_1\bar{\gamma}_2^3 - 108\gamma_1\bar{\gamma}_2\gamma_4 - 3\gamma_1^3\gamma_4 - 2\gamma_1^2\bar{\gamma}_2^2 \\ &\quad + 243\gamma_3\gamma_4 - 18\gamma_1\gamma_3^2 + 27\bar{\gamma}_2\gamma_2\gamma_3 - 9\gamma_1\gamma_2\gamma_4)u^3 \\ &\quad + 6(3\gamma_1^2\bar{\gamma}_2\gamma_4 - 6\bar{\gamma}_2^4 - 2\gamma_1^2\bar{\gamma}_2^3 + 6\gamma_1\bar{\gamma}_2^2\gamma_3 + 108\bar{\gamma}_2^2\gamma_4 \\ &\quad - 21\bar{\gamma}_2\gamma_2\gamma_4 + 27\gamma_1\gamma_3\gamma_4 - 15\bar{\gamma}_2\gamma_3^2 - 324\gamma_4^2)u^2 \\ &\quad + 3(10\gamma_1\bar{\gamma}_2^4 - 45\gamma_1\bar{\gamma}_2^3\gamma_4 + 27\gamma_1\gamma_4^2 - 15\bar{\gamma}_2^3\gamma_3 + 54\bar{\gamma}_2\gamma_3\gamma_4)u \\ &\quad + 2(45\bar{\gamma}_2^3\gamma_4 - 6\bar{\gamma}_2^5 - 81\bar{\gamma}_2\gamma_4^2)], \\ 729a_6 &= (4\gamma_1^6 - 36\gamma_1^4\gamma_2 - 18\gamma_1^4\bar{\gamma}_2 + 54\gamma_1^3\gamma_3 - 162\gamma_1^2\gamma_4 + 81\gamma_1^2\gamma_2^2 \\ &\quad + 81\gamma_1^2\bar{\gamma}_2\gamma_2 - 243\gamma_1\gamma_2\gamma_3 + 729\gamma_2\gamma_4)u^6 \\ &\quad + 3(6\gamma_1^5\gamma_3 - 4\gamma_1^5\bar{\gamma}_2 + 24\gamma_1^3\bar{\gamma}_2\gamma_2 + 21\gamma_1^3\bar{\gamma}_2^2 + 90\gamma_1^3\gamma_4 - 135\gamma_1^2\bar{\gamma}_2\gamma_3 \\ &\quad - 27\gamma_1^2\gamma_2\gamma_3 - 27\gamma_1\bar{\gamma}_2\gamma_4 - 27\gamma_1\bar{\gamma}_2\gamma_2^2 + 108\gamma_1\gamma_3^2 - 351\gamma_1\gamma_2\gamma_4 \\ &\quad + 162\bar{\gamma}_2\gamma_2\gamma_3)u^5 \\ &\quad + 3(-\gamma_1^4\bar{\gamma}_2^2 - 3\gamma_1^4\gamma_4 + 6\gamma_1^3\bar{\gamma}_2\gamma_3 - 9\gamma_1^2\gamma_3^2 - 9\gamma_1^2\bar{\gamma}_2^3 - 27\gamma_1^2\bar{\gamma}_2\gamma_4 \\ &\quad + 9\gamma_1^2\gamma_2\gamma_4 + 81\gamma_1\bar{\gamma}_2^2\gamma_3 - 27\gamma_1\bar{\gamma}_2\gamma_2\gamma_3 - 108\bar{\gamma}_2\gamma_3^2 + 27\bar{\gamma}_2^2\gamma_4 \\ &\quad + 54\gamma_2\gamma_3^2 + 27\bar{\gamma}_2\gamma_2\gamma_4 + 27\gamma_2^2\gamma_4)u^4 \\ &\quad + (26\gamma_1^3\bar{\gamma}_2^3 - 117\gamma_1^2\bar{\gamma}_2^2\gamma_3 - 72\gamma_1\bar{\gamma}_2^4 + 162\gamma_1\bar{\gamma}_2^2\gamma_4 - 9\gamma_1\bar{\gamma}_2\gamma_2\gamma_4 \\ &\quad + 225\gamma_1\bar{\gamma}_2\gamma_3^2 + 54\bar{\gamma}_2^3\gamma_3 - 81\gamma_2\gamma_3\gamma_4 - 162\gamma_3^3)u^3 \\ &\quad + 3(3\gamma_1^2\bar{\gamma}_2^2\gamma_4 - \gamma_1^2\bar{\gamma}_2^4 + 3\gamma_1^2\gamma_4^2 + 3\gamma_1\bar{\gamma}_2^2\gamma_3 - 30\gamma_1\bar{\gamma}_2\gamma_3\gamma_4 \\ &\quad - 6\bar{\gamma}_2^2\gamma_3^2 + 57\gamma_3^3\gamma_4 + 15\gamma_2\gamma_4^2 - 45\bar{\gamma}_2^3\gamma_4 + 12\bar{\gamma}_2^5)u^2 \\ &\quad + 3(24\gamma_1\bar{\gamma}_2^3\gamma_4 - 4\gamma_1\bar{\gamma}_2^5 - 27\gamma_1\bar{\gamma}_2\gamma_4^2 + 6\bar{\gamma}_2^4\gamma_3 - 27\bar{\gamma}_2^2\gamma_3\gamma_4)u \\ &\quad + (4\bar{\gamma}_2^6 - 36\bar{\gamma}_2^4\gamma_4 + 81\bar{\gamma}_2^2\gamma_4^2). \end{aligned} \quad (6)$$

We can change to symmetric functions at any time by means of the relations

$$\left. \begin{aligned} p_2 &= -2a_2, \\ p_3 &= -3a_3, \\ p_4 &= -4a_4 + 2a_2^2, \\ p_5 &= -5a_2a_3 + 5a_3, \\ p_6 &= -6a_6 + 6a_2a_4 + 3a_3^2 - 2a_2^3. \end{aligned} \right\} \quad (7)$$

To determine the cubic curves a, \dots, f on the prepared set (3), solve the five relations (*loc. cit.*, p. 168) with $\Sigma a = 0$, thus getting the following values, where we write in order the coefficients of $x_0x_1^2, x_0^2x_1, x_1^2x_2, x_1x_2^2, x_0^2x_2, x_0x_2^2, x_0x_1x_2$:

$$\left. \begin{aligned} \frac{a}{2} &= u^2 - ux - uz, -u^2 + uy + ut, xz + uz - ux, xt - xz - yz, \\ &\quad ut - uy - ty, yz - tx + ty, 2(ux - ut) \\ \frac{b}{2} &= -u^2 + ux - uz, u^2 + ut - uy, ux + uz - xz, xz - yz - xt, \\ &\quad uy - ut - ty, xt + ty - yz, 2(yz - ux), \\ \frac{c}{2} &= -u^2 - ux + uz, u^2 + uy - ut, xz + ux - uz, yz - xz - xt, \\ &\quad ty - uy - ut, yz + tx - ty, 2(ut - yz), \\ \frac{d}{2} &= -u^2 + ux + uz, u^2 - uy - ut, uz - ux - xz, xz - yz + xt, \\ &\quad ty - uy + ut, yz - xt - yt, 2(uy - uz) \\ \frac{e}{2} &= u^2 - ux + uz, -u^2 + uy - ut, ux - uz - xz, xz + yz - xt, \\ &\quad uy + ut - ty, ty - yz - tx, 2(xt - uy), \\ \frac{f}{2} &= u^2 + ux - uz, -u^2 + ut - uy, xz - ux - uz, yz + tx - xz, \\ &\quad ty + uy - ut, xt - yz - ty, 2(uz - xt). \end{aligned} \right\} \quad (8)$$

Combining these with the values of \bar{a}, \dots, \bar{f} , given by (4), we find immediately the functions K_2, \dots, K_5 .

From the values of p_2, \dots, p_5 , given by (6), we find that $p_2^2 - 4p_4$ or $a_2^2 - 4a_4$, and $12p_5 - 5p_2p_3$ or $a_2a_3 - 2a_5$ contain u^2 as a factor. If, then,

$$q_4 = p_2^2 - 4p_4, \quad q_5 = 12p_5 - 5p_2p_3,$$

we find

$$\left. \begin{aligned} q_4 &= u^2 [(\bar{\gamma}_2^2 - 4\gamma_4)u^2 + 2(2\gamma_1\gamma_4 - \bar{\gamma}_2\gamma_3)u + \gamma_3^2 - 4\gamma_2\gamma_4], \\ q_5 &= 2430u^2 [(3\bar{\gamma}_2\gamma_3 - 2\gamma_1\gamma_4 - \gamma_1\bar{\gamma}_2^2)u^3 \\ &\quad + (2\bar{\gamma}_2^2 - 8\bar{\gamma}_2\gamma_4 - \gamma_1\bar{\gamma}_2\gamma_3 + 5\gamma_1^2\gamma_4 - 12\gamma_2\gamma_4)u^2 \\ &\quad + (3\bar{\gamma}_2\gamma_2\gamma_3 - \bar{\gamma}_2^2\gamma_3 - 7\gamma_1\bar{\gamma}_2\gamma_4 - 2\gamma_1\gamma_2\gamma_4 - \gamma_1\gamma_3^2 + 18\gamma_3\gamma_4)u \\ &\quad + (6\bar{\gamma}_2^2\gamma_4 - \bar{\gamma}_2\gamma_3^2 - 2\bar{\gamma}_2\gamma_2\gamma_4 + 3\gamma_1\gamma_3\gamma_4 - 24\gamma_1^2)]]. \end{aligned} \right\} \quad (9)$$

Calculating the terms free of u in the functions K_2, \dots, K_5 , I find

$$\left. \begin{aligned} K_2 &= 36\gamma_4(xt-yz)(x_1x_2^2-x_0x_2^2), \\ K_3 &= 0, \\ K_4 &= 108\gamma_4(\bar{\gamma}_2^2-3\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2), \\ K_5 &= 36\gamma_4(-2\bar{\gamma}_2^3-81\bar{\gamma}_2\gamma_4)(xt-yz)(x_1x_2^2-x_0x_2^2); \end{aligned} \right\} \quad (10)$$

hence, the combinations $K'_4 = 9p_2K_2 - 4K_4$, $K'_5 = 9(p_3 - 20\bar{\gamma}_2\gamma_4)K_2 - 2K_5$, contain u as a factor. It follows from the value of K'_5 that it could not be used to advantage in expressing the linear covariants.

An interesting particular case occurs when the last two points of (3) are apolar to the pencil of conics on the first four, $x_2x_3 = x_3x_1 = x_1x_2$; i. e., when two of the points are a corresponding pair in the quadratic Cremona involution with fixed points at the other four. This requires that $xt + yz = u(y+t) = u(x+z)$ whence from (4) $\bar{a} + \bar{d} = \bar{b} + \bar{f} = \bar{c} + \bar{e} = \lambda$. Since $\Sigma \bar{a} = 0$, $\lambda = 0$ and, therefore, $p_3 = \Sigma \bar{a}^3 = 0$ and $p_5 = \Sigma \bar{a}^5 = 0$. If, conversely, $p_3 = p_5 = 0$ and if five points of P_6^2 be fixed there are fifteen possible positions for the sixth. These all are accounted for by choosing any four of the five as fixed points or any three of the five as fixed points. Hence,

The necessary and sufficient condition that the invariants of P_6^2 of odd degree (other than the skew invariant $d_2\sqrt{d}$) vanish is that two of the points be apolar to the pencil of conics on the other four.

Section 3.

Let the cubic surface be given by

$$U = (\alpha x)^3 = (\beta x)^3 = \dots = (\delta x)^3, \text{ or } = x_1^3 + x_2^3 + \dots + x_6^3 = 0, \quad (11)$$

for the Cremona form, where the variables are subject to the relations

$$\left. \begin{aligned} x_1 + x_2 + \dots + x_6 &= 0, \\ \bar{a}x_1 + \bar{b}x_2 + \dots + \bar{f}x_6 &= 0. \end{aligned} \right\} \quad (12)$$

The Hessian is the locus of points whose polar quadrics are cones and, therefore, obtained by writing the discriminant of a polar quadric.

According to the principle of transference of Clebsch it is given by

$$H = \frac{1}{24} |\alpha\beta\gamma\delta\eta\zeta|^2 (\alpha x)(\beta x)(\gamma x)(\delta x) \text{ or } = \sum_{(56)}^{15} x_1x_2x_3x_4\bar{56}^2 \quad (13)$$

for the Cremona form, where $\bar{ij} = \begin{vmatrix} \eta_i & \eta_j \\ \zeta_i & \zeta_j \end{vmatrix}$, and since $\eta_1, \dots, \eta_6 = 1, \dots, 1$, $\zeta_1, \dots, \zeta_6 = \bar{a}, \dots, \bar{f} = b_1, \dots, b_6$, we have $\bar{ij} = b_i - b_j$. The super- and subscript of Σ refer to the fifteen terms obtained by choosing the pair 56 from 1, ..., 6.

The contravariant S , as given by the Clebsch transference principle, is

$$S = \frac{1}{6} |\alpha\beta\gamma\eta\zeta\xi| |\alpha\beta\delta\eta\zeta\xi| |\alpha\gamma\delta\eta\zeta\xi| |\beta\gamma\delta\eta\zeta\xi| \text{ or } = 4 \sum_{(56)}^{15} [\overline{156} \overline{256} \overline{356} \overline{456}] \quad (14)$$

for the Cremona form, where

$$\overline{ijk} = \begin{vmatrix} \xi_i & \xi_j & \xi_k \\ \eta_i & \eta_j & \eta_k \\ \zeta_i & \zeta_j & \zeta_k \end{vmatrix}.$$

By expanding and collecting the coefficients of ξ we have

$$\begin{aligned} \frac{1}{4} S = & \sum_{(1)}^6 \xi_1^4 \sum_{(2)}^5 \overline{23} \overline{24} \overline{25} \overline{26} + \sum_{(1)(2)}^{30} \xi_1^3 \xi_2 [\sum_{(3)}^4 \overline{31} \overline{24} \overline{25} \overline{26} + \sum_{(8)}^4 \overline{13} \overline{34} \overline{35} \overline{36}] \\ & + \sum_{(56)}^{15} \xi_5^2 \xi_6^2 \sum_{(12)}^6 \overline{16} \overline{26} \overline{35} \overline{45} + \sum_{(12)(3)}^{60} \xi_1 \xi_2 \xi_3^2 [\sum_{(4)}^3 \overline{34}^2 \overline{45} \overline{46} + \sum_{(1)}^2 \overline{31} \sum_{(4)}^3 \overline{34} \overline{15} \overline{16}] \\ & + \sum_{(1234)}^{15} \xi_1 \xi_2 \xi_3 \xi_4 [\overline{56}^4 + \sum_{(1)}^4 \sum_{(5)}^2 \overline{15}^3 \overline{56} - \sum_{(12)}^6 \sum_{(5)}^2 \overline{12}^2 \overline{15} \overline{26}]. \end{aligned}$$

Let us indicate the coefficients of S as follows:

$$\frac{1}{4} S = \sum_{(1)}^6 S_1 \xi_1^4 + \sum_{(1)}^6 \sum_{(2)}^5 S_{1,2} \xi_1^3 \xi_2 + \sum_{(12)}^{15} S_{12} \xi_1^2 \xi_2^2 + \sum_{(1)}^6 \sum_{(23)}^{10} S_{1,23} \xi_1^2 \xi_2 \xi_3 + \sum_{(1234)}^{15} S_{1234} \xi_1 \xi_2 \xi_3 \xi_4.$$

We find that these coefficients expressed in terms of a_2, \dots, a_6 , the elementary symmetric functions of b_1, \dots, b_6 are:

$$\left. \begin{aligned} S_1 &= -15a_4 + 4a_2^2 + 9a_3b_1 - 6a_2b_1^2 - 9b_1^4, \\ S_{1,2} &= 12a_4 - 4a_2^2 + 3a_3\{1\} - 2a_2\{1, 2\} - 6\{0, 1, 0\} - 6b_2[a_4 + 2b_1b_2^3], \\ S_{12} &= 6a_4 - 9a_3\{1\} + 2a_2\{5, -2\} + 10\{1, -2, 1\}, \\ S_{1,23} &= -8a_4 + 4a_2^2 - a_3[10b_1 + \{1\}] + 2a_2[b_1\{4\} + \{3, -2\}] \\ &\quad + 2[2b_1\{4, -9\} + \{2, -3, 0\}], \\ S_{1234} &= 8a_2^2 + 18a_3\{1\} + 12a_2\{1, -2\} - 4\{2, -7, -4\}. \end{aligned} \right\} \quad (15)$$

Here $\{\dots\}$ refers to the coefficients of polynomials in ρ_1, ρ_2 the symmetric combinations of the two isolated letters.

Identities among these which are sometimes useful are:

$$\left. \begin{aligned} \{1, -4, 3\} + a_2\{1, -2\} - a_3\{1, -1\} + a_4\{1\} - a_5 &= 0, \\ [b_1^4 + b_1^3\{1\} + b_1^2\{1, -1\} + b_1\{1, -2\} + \{1, -3, 1\}] \\ + a_2[b_1^2 + b_1\{1\} + \{1, -1\}] - a_3[b_1 + \{1\}] + a_4 &= 0. \end{aligned} \right\} \quad (16)$$

The coefficients of S are connected by the relations

$$\begin{aligned} 4S_1\eta_1 + \sum_{(2)}^5 S_{1,2}\eta_2 &= 0, \\ 3S_{1,2}\eta_1 + 2S_{12}\eta_2 + \sum_{(3)}^4 S_{1,23}\eta_3 &= 0, \\ 2(S_{1,23}\eta_1 + S_{2,31}\eta_2 + S_{3,12}\eta_3) + S_{1234}\eta_4 + S_{1235}\eta_5 + S_{1236}\eta_6 &= 0. \end{aligned}$$

Hence, $\eta_1, \dots, \eta_6 = 1, \dots, 1$, or b_1, \dots, b_6 , also $\sum_{(2)}^5 S_{2,1} = \sum_{(2)}^5 S_{1,2} = -4S_1$.

A numerical check is furnished by taking $\begin{Bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ 1 & 1 & 1 & 1 & -2 & -2 \end{Bmatrix}$, then $a_2 = -6$, $a_3 = -4$, $a_4 = 9$, $a_5 = 12$, $a_6 = 4$, while $S_5 = S_6 = 81$, $S_{55} = 18 \cdot 27$, $S_{5,6} = S_{6,5} = -18 \cdot 18$, and all the other coefficients S vanish.

By operating with S on H (in operating with a form of order n on a form of class $m > n$ we remove the factor $m(m-1) \dots (m-n+1)$ after differentiation) we get the first invariant

$$\begin{aligned} I_1 &= \sum_{(56)}^{15} 56^6 - 2 \sum_{(1)}^6 \sum_{(23)}^{10} 12^3 13^3 - 2 \sum_{(12)(34)}^{45} 12^2 34^2 [13 \overline{24} + 14 \overline{23}] \\ &= -15 \sum b_1^6 + 18 \sum b_1^5 b_2 - 9 \sum b_1^4 b_2^2 - 18 \sum b_1^4 b_2 b_3 \\ &\quad + 12 \sum b_1^3 b_2^3 + 24 \sum b_1^3 b_2 b_3 b_4 + 18 \sum b_1^2 b_2^2 b_3^2 - 12 \sum b_1^2 b_2^2 b_3 b_4, \end{aligned}$$

and in terms of the symmetric functions a_2, \dots, a_6 we find

$$I_1 = 24[4a_2^3 - 3a_3^2 - 16a_2 a_4 + 12a_6]. \quad (17)$$

By operating with S on U^2 we get the first covariant quadric, i. e., a $C_{6,2,0}$. Denoting this quadric by Q , we find

$$5Q = \sum_{(1)}^6 (20S_1 + \sum_{(2)}^5 2S_{2,1})x_1^2 + 4 \sum_{(12)}^{15} S_{12}x_1x_2 = \sum_{(1)}^6 Q_{11}x_1^2 + \sum_{(12)}^{15} Q_{12}x_1x_2, \quad (18)$$

where

$$\begin{aligned} Q_{11} &= 12S_1, & Q_{12} &= 4S_{12}, \\ (S_1 \text{ and } S_{12} \text{ are given by (15)}). \end{aligned} \quad (19)$$

By operating with Q on S we get the contravariant quadric $C_{10,0,2}$. Denoting this quadric by q we find

$$5 \cdot 3q = \sum_{(1)}^6 q_{11}\xi_1^2 + \sum_{(12)}^{15} q_{12}\xi_1\xi_2, \quad (20)$$

where if $q_4 = a_2^2 - 4a_4$ and $q_5 = a_2 a_6 - 2a_5$,

$$\left. \begin{aligned} \frac{1}{72}q_{11} &= -3 \cdot 32q_4 b_1^4 + 3 \cdot 8q_5 b_1^3 - 3 \cdot 2[I_1 + 8a_2 q_4]b_1^2 \\ &\quad + 3 \cdot 2[5a_2 q_5 + 17a_3 q_4]b_1 + [5a_2 I_1 + 15a_3 q_5 + 27q_4^2 - 15a_2^2 q_4], \\ \frac{1}{32}q_{12} &= -32a_2 a_6 - 135a_3 a_5 - 216a_4^2 + 198a_2^2 a_4 - 36a_2^4 \\ &\quad + 135a_2 a_5 \{1\} - 108a_3 a_4 \{1\} + 27a_2^2 a_3 \{1\} + 12a_6 \{ -43, 33 \} \\ &\quad + 2a_2 a_4 \{144, -47\} + 81a_3^2 \{0, 1\} - 36a_2^3 \{2, 1\} + 6a_5 \{45, -4\} \\ &\quad - 248a_2 a_3 \{0, 1\} + 36a_4 \{6, -9, -14\} + 2a_2^2 \{ -27, -11, 173 \} \\ &\quad + 6a_3 \{0, -27, 61\} + 2a_2 \{0, 16, 93, -83\} + 12\{0, 0, 10, 13, -6\}. \end{aligned} \right\} \quad (21)$$

By operating with q on U we get the first linear covariant $C_{11,1,0}$. Denoting this covariant by L_1 we find $L_1 = q_{11}x_1 + q_{22}x_2 + \dots + q_{66}x_6$, or in terms of the functions (2),

$$L_1 = -16q_4K_4 + 4q_5K_5 + [-I_1 - 8a_2q_4]K_2. \quad (22)$$

By operating with q on Q we get the second invariant I_2 , where

$$5^2 \cdot 6 I_2 = 2 \sum_{(1)}^6 q_{11} Q_{11} + \sum_{(12)}^{15} q_{12} Q_{12}.$$

By another method I have found a second invariant to be

$$I'_2 = 2^3 q_5^2 q_2 + 3 \cdot 5 \cdot 2^4 q_5 q_4 q_3 - 2 \cdot 3^2 \cdot 5^2 q_4^2 - 2 \cdot 3 \cdot 5^2 q_4^2 q_2^2 + 3 \cdot 5^2 I_1 q_4 q_2; \quad (23)$$

hence, we must find $I_2 = c_0 I_1^2 + c_1 I'_2$, where c_0 and c_1 are numerical constants.

If we denote by M the mixed concomitant obtained by writing the plane equation of the polar quadric of any point with respect to the cubic surface, we have, according to the principle of transference of Clebsch,

$$\left. \begin{aligned} M &= \frac{1}{6} |\alpha\beta\gamma\eta\zeta\xi| (ax)(\beta x)(\gamma x), \text{ or } = \sum_{(123)}^{20} x_1 x_2 x_3 \overline{456^2}, \\ &= \sum_{(123)}^{20} x_1 x_2 x_3 \left(\sum_{(4)}^8 u_4^2 \overline{56^2} - 2 \sum_{(56)}^8 u_5 u_6 \overline{4546} \right) \end{aligned} \right\} \quad (24)$$

for the Cremona form.

By operating on this with $\frac{5}{4}Q = \sum_{(1)}^5 3S_1 x_1^2 + \sum_{(12)}^{15} S_{12} x_1 x_2$, we get a $C_{9,2,0}$.

Denoting this by N we have

$$N = \sum_{(123)}^{20} x_1 x_2 x_3 \left[\sum_{(4)}^8 (3S_4 \overline{56^2} - S_{56} \overline{4546}) \right] = \sum_{(123)}^{20} C_{123} x_1 x_2 x_3, \quad (25)$$

where we find

$$\begin{aligned} C_{123} &= 24q_4(\sigma_1^2 - 3\sigma_2) + 36a_3(\sigma_1\sigma_2 - 9\sigma_3) + 72a_2(3\sigma_1\sigma_3 - \sigma_2^2) \\ &\quad + 3[19(4\sigma_1^3\sigma_3 - \sigma_1^2\sigma_2^2) + 28\sigma_2^3 - 150\sigma_1\sigma_2\sigma_3 + 81\sigma_3^2]. \end{aligned} \quad (26)$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions of b_4, b_5, b_6 .

The result of operating with N on S is a $C_{13,0,1}$. Denoting this by C , we have

$$C = \sum_{(1)}^6 \xi_1 \left[\sum_{(56)}^{10} (S_{1234} C_{234} + 2S_{1,56} C_{156}) \right] = C_1 \xi_1 + \dots + C_6 \xi_6. \quad (27)$$

Multiplying $S_{1,56}$ and C_{156} together, summing for \sum^{10} and indicating by $(ijkl)$ the symmetric sum for five things of $\sum b_2^i b_3^j b_4^k b_5^l b_6^i$, we get

$$\begin{aligned}
 \sum^{10} 2S_{1,56} C_{156} = & 6 \{ 24q_4[4a_2^2 - 8a_4 - 10a_3b_1][2(2) - (11)] \\
 & + \{ 24q_4[-a_3 + 8a_2b_1] + 36a_3[4a_2^2 - 8a_4 - 10a_3b_1] \} [(21) - 2(111)] \\
 & + 48a_3[-a_3 + 8a_2b_1][(211) - 6(1111)] \\
 & \quad + 24a_2[4a_2^2 - 8a_4 - 10a_3b_1][(211) - 3(22)] \\
 & + 32a_2q_4[9(22) - (211) - 12(1111)] \\
 & + [4a_2^2 - 8a_4 - 10a_3b_1][38(411) - 57(42) \\
 & \quad + 10(321) - 30(33) - 30(222)] \\
 & + 32q_4b_1[12(32) - 8(311) + 6(221) - 9(2111)] \\
 & + 144a_2a_3[2(221) - (2111) - 40(11111)] \\
 & \quad + 24a_2[-a_3 + 8a_2b_1][3(2111) - 2(221)] \\
 & + 96a_2^2[(2211) - 9(222) + 12(21111)] \\
 & \quad + 16 \cdot 12a_3b_1[2(321) - 6(3111) + 3(2211) - 18(21111)] \\
 & + 16q_4[6(42) - 4(411) + 5(321) - 15(3111) \\
 & \quad + 18(222) - 6(2211)] \} \quad (28) \\
 & + 48a_3[2(421) - 6(4111) + 5(3211) - 60(31111) \\
 & \quad + 9(2221) - 18(22111)] \\
 & + 96a_2b_1[4(3211) - 8(322) + 18(22111) - 9(2221)] \\
 & + 2[-a_3 + 8a_2b_1][19 \cdot 12(31111) - 19(421) \\
 & \quad + 20(3211) - 20(331) - 30(2221)] \\
 & + 48a_2[2(4211) - 4(422) + 15(32111) - 5(3221) \\
 & \quad + 18(22211) - 36(2222)] \\
 & + 4a_3[19\{(4211) - 6(422) + 24(41111)\} + 10\{3(3221) - 3(332) \\
 & \quad - 18(2222) + 6(32111) - 2(3311) - 6(22211)\}] \\
 & + 38[8(4411) - 8(442) + 30(43111) - 5(4321) \\
 & \quad + 12(42211) - 18(4222)] \\
 & + 20[2(4321) - 4(433) - 6(4222) + 20(33211) - 15(3331) \\
 & \quad + 3(32221) - 6(3322) - 180(22222)] \}.
 \end{aligned}$$

In order to get $\sum^{10} S_{1234} C_{234}$, we write

$$\begin{aligned}
 S_{1234} &= 2[4a_2^2 + 9a_3(b_5 + b_6) + 6a_2(b_5^2 + b_6^2) - 4(b_5^4 + b_6^4) - 2b_5b_6(b_5^2 + b_6^2) + 12b_5^2b_6^2], \\
 C_{234} &= 3[8q_4b_1^2 + (-8q_4b_1 + 12a_3b_1^2)(b_5 + b_6) + (8q_4 + 12a_3b_1 - 24a_2b_1^2 \\
 & \quad - 19b_1^4)(b_5^2 + b_6^2) + (-8q_4 - 72a_3b_1 + 24a_2b_1^2 + 38b_1^4)b_5b_6 - 10b_1^2(b_5^3 + b_6^3) \\
 & \quad + (12a_3 + 24a_2b_1 + 10b_1^3)(b_5 + b_6)b_5b_6 - 19b_1^2(b_5^4 + b_6^4) + 10b_1^2b_5b_6(b_5^2 + b_6^2) \\
 & \quad + (-24a_2 - 30b_1^2)b_5^2b_6^2 + 38b_1b_5b_6(b_5^2 + b_6^2) + 10b_1b_5^2b_6^2(b_5 + b_6) \\
 & \quad - 19b_5^2b_6^2(b_5^2 + b_6^2) - 10b_5^2b_6^3].
 \end{aligned}$$

$$\begin{aligned}
\therefore \sum_{1234}^{10} C_{234} = & 6 \{ 320 a_2^2 q_4 b_1^2 + [72 a_3 q_4 b_1^2 + 48 a_3 a_2^2 b_1^2 - 32 a_2^2 q_4 b_1] 4(1) \\
& + [-76 a_2^2 b_1^4 + 12 (-8 a_2^3 + 9 a_3^2 + 4 a_2 q_4) b_1^2 + 24 (2 a_2^2 a_3 - 3 a_3 q_4) b_1 \\
& \quad + 32 a_2^2 q_4] 4(2) \\
& + [8 \cdot 19 a_2^2 b_1^4 + 24 (9 a_3^2 + 4 a_2^3) b_1^2 - 144 (a_3 q_4 + 2 a_2^2 a_3) b_1 - 32 a_2^2 q_4] (11) \\
& + [-9 \cdot 19 a_3 b_1^4 - 40 a_2^2 b_1^3 - 144 a_2 a_3 b_1^2 + 12 (9 a_3^2 - 4 a_2 q_4) b_1 \\
& \quad + 72 a_3 q_4] 4(3) \\
& + [9 \cdot 19 a_3 b_1^4 + 40 a_2^2 b_1^3 + 72 a_2 a_3 b_1^2 + 12 (8 a_2^3 + 9 a_3^2 - 4 a_2 q_4) b_1 \\
& \quad + 48 a_2^2 a_3] (21) \\
& + [-6 \cdot 19 a_2 b_1^4 - 90 a_3 b_1^3 - 4 (55 a_2^2 + 8 q_4) b_1^2 + 72 a_2 a_3 b_1 + 48 a_2 q_4] 4(4) \\
& + [12 \cdot 19 a_2 b_1^4 + 8 (23 a_2^2 - 2 q_4) b_1^2 - 24 \cdot 9 a_2 a_3 b_1 + 12 (9 a_3^2 - 4 a_2 q_4)] (31) \\
& + [-12 \cdot 19 a_2 b_1^4 + 180 a_3 b_1^3 + 24 (-17 a_2^2 + 4 q_4) b_1^2 + 24 \cdot 24 a_2 a_3 b_1 \\
& \quad + 24 (9 a_3^2 - 4 a_2^3 + 4 a_2 q_4)] (22) \\
& + [-60 a_2 b_1^3 - 3 \cdot 73 a_3 b_1^2 + 32 q_4 b_1] 4(5) \\
& + [60 a_3 b_1^3 - 9 \cdot 17 a_3 b_1^2 + (8 \cdot 37 a_2^2 + 48 q_4) b_1 + 72 a_2 a_3] (41) \\
& + [-60 a_3 b_1^2 + 8 (23 a_2^2 - 10 q_4) b_1 - 144 a_2 a_3] (32) \\
& + [4 \cdot 19 b_1^4 - 18 a_2 b_1^3 - 48 a_3 b_1 - 32 q_4] 4(6) \\
& + [-6 \cdot 19 b_1^4 + 12 a_2 b_1^3 + 606 a_3 b_1 + 16 q_4] (51) \\
& + [-12 \cdot 19 b_1^4 - 6 \cdot 89 a_2 b_1^3 + 24 \cdot 28 a_3 b_1 + 20 (4 q_4 - 11 a_2^2)] (42) \\
& + [28 \cdot 19 b_1^4 + 12 \cdot 42 a_2 b_1^3 - 6 \cdot 122 a_3 b_1 - 8 (16 q_4 + 5 a_2^2)] (33) \\
& + [40 b_1^3] 4(7) + [-20 b_1^3 + 12 \cdot 11 a_2 b_1 - 48 a_3] (61) \\
& + [-180 b_1^3 - 84 a_2 b_1 - 9 \cdot 27 a_3] (52) \\
& + [160 b_1^3 + 48 \cdot 11 a_2 b_1 - 3 \cdot 47 a_3] (43) \\
& + [4 \cdot 19 b_1^2] 4(8) + [-2 b_1^2] (71) + [-16 \cdot 8 b_1^2 - 18 a_2] (62) \\
& + [2 \cdot 89 b_1^2 - 12 a_2] (53) + [-4 \cdot 62 b_1^2 - 129 a_2] (44) \\
& - 8 \cdot 19 b_1 (81) - 4 \cdot 29 b_1 (72) + 12 \cdot 33 b_1 (63) - 4 \cdot 32 b_1 (54) \\
& + 4 \cdot 19 (82) + 78 (73) - 4 \cdot 33 (64) - 4 \cdot 11 (55). \} \quad (29)
\end{aligned}$$

Using brackets for symmetric functions of six things, we have

$$\left. \begin{aligned}
(i) &= [i] - b_1^i, \\
(ii) &= [ii] - b_1^i [i] + b_1^{2i}, \\
(ij) &= [ij] - b_1^i [j] - b_1^j [i] + 2 b_1^{i+j}, \text{ etc.;}
\end{aligned} \right\} \quad (30)$$

thus, we get C_1 in final form by making these substitutions in (28) and (32) and collecting.

The polar form of (18) is $\frac{1}{2} [\sum_{(1)}^6 x_1 (2 Q_{11} y_1 + \sum_{(2)}^5 Q_{12} y_2)]$, and by operating with this on (20) we get a collineation $C_{10,1,1}$. Denoting this by K , we have

$$2K = \sum_{(1)}^6 \xi_1 [x_1 (4 q_{11} Q_{11} + \sum_{(2)}^5 q_{12} Q_{12}) + \sum_{(2)}^5 x_2 (2 q_{12} Q_{22} + 2 q_{11} Q_{12} + \sum_{(3)}^4 q_{13} Q_{23})]. \quad (31)$$

If this collineation be written as

$$2^7 K = p_{11} \xi_1 + p_{22} \xi_2 + \dots + p_{66} \xi_6, \quad (32)$$

we find

$$p_{11} = \left. \begin{aligned} & x_1 [d_1 + b_1 d_2 + b_1^2 d_3 + b_1^3 d_4 + b_1^4 d_5 + b_1^5 d_6] \\ & + K_2 [f_{2,0} + b_1 f_{2,1} + b_1^2 f_{2,2} + b_1^3 f_{2,3} + b_1^4 f_{2,4} + b_1^5 f_{2,5}] \\ & + K_3 [f_{3,0} + b_1 f_{3,1} + b_1^2 f_{3,2} + b_1^3 f_{3,3} + b_1^4 f_{3,4} + b_1^5 f_{3,5}] \\ & + K_4 [f_{4,0} + b_1 f_{4,1} + b_1^2 f_{4,2} + b_1^3 f_{4,3} + b_1^4 f_{4,4} + b_1^5 f_{4,5}] \\ & + K_5 [f_{5,0} + b_1 f_{5,1} + b_1^2 f_{5,2} + b_1^3 f_{5,3} + b_1^4 f_{5,4} + b_1^5 f_{5,5}], \end{aligned} \right\} \quad (33)$$

where

$$\left. \begin{aligned} d_1 &= 2^4 \cdot 3^2 \cdot 53 I_1^2 + 2^4 \cdot 3^3 \cdot 53 a_2^2 I_1 - 5 \cdot 10181 a_2 q_4 I_1 + 2^4 \cdot 3^4 \cdot 5 q_4^3 \\ &\quad + 2^2 \cdot 13 \cdot 1609 a_2^2 q_4^2 - 3 \cdot 7 \cdot 2911 a_2 a_3 q_4 + 5^2 a_2^3 I_1 + 3 \cdot 5^2 a_2^3 a_3^2 - 2^2 \cdot 5^2 a_2^4 q_4, \\ d_2 &= 6(2^4 \cdot 3 \cdot 359 q_5 I_1 - 2^4 \cdot 3^2 \cdot 53 a_2 a_3 I_1 - 3^2 \cdot 5^2 \cdot 257 a_2 q_4 q_5 + 2^7 \cdot 3^2 \cdot 5^2 a_2^2 q_5 \\ &\quad + 5^2 a_2^3 q_5 + 2^4 \cdot 3^4 \cdot 5 a_3 q_4^2 + 3 \cdot 5 \cdot 1439 a_2^2 a_3 q_4 - 5^2 a_2^4 a_3), \\ d_3 &= 3(-2^4 \cdot 3^2 \cdot 53 q_4 I_1 + 2^4 \cdot 461 a_2^2 I_1 + 2^9 \cdot 3^2 \cdot 5^2 q_5^2 - 2^4 \cdot 3 \cdot 2281 a_2 a_3 q_5 \\ &\quad + 14153 a_2 q_4^2 + 2^4 \cdot 3^5 \cdot 5 a_2^2 q_4 - 2 \cdot 12097 a_2^3 q_4 - 2^4 \cdot 3^2 \cdot 7^2 a_2^2 a_3^2 + 5 \cdot 101 a_2^5), \\ d_4 &= 12(-2^4 \cdot 3^2 \cdot 53 a_3 I_1 - 2^4 \cdot 3 \cdot 7 \cdot 37 q_4 q_5 + 2^2 \cdot 3 \cdot 1049 a_2^2 q_5 \\ &\quad + 5 \cdot 3473 a_2 a_3 q_4), \\ d_5 &= 12(2^3 \cdot 3^2 \cdot 83 a_2 I_1 - 2^4 \cdot 3^2 \cdot 359 a_3 q_5 - 2^4 \cdot 3 \cdot 41 q_4^2 - 3 \cdot 5539 a_2^2 q_4 + 25 a_2^4), \\ d_6 &= 288(1621 a_2 q_5 + 2^3 \cdot 3^2 a_3 q_4), \\ f_{2,0} &= 6(17 q_4 I_1 - 563 a_2^2 I_1 - 3^2 \cdot 5 q_5^2 - 2^3 \cdot 3^2 \cdot 5 a_2 a_3 q_5 + 2^3 \cdot 59 a_2 q_4^2 \\ &\quad - 3 \cdot 487 a_2^3 q_4 + 2^3 \cdot 5 \cdot 89 a_2^3 q_4 - 2^2 \cdot 3 \cdot 143 a_2^2 a_3^2 - 2^4 \cdot 3 \cdot 5 a_2^5), \\ f_{3,0} &= 18(421 a_3 I_1 + 2 \cdot 3 \cdot 23 a_2^2 q_5 - 2 \cdot 5 \cdot 127 a_2 a_3 q_4 + 2^3 \cdot 3 \cdot 53 a_3^3), \\ f_{2,1} &= 6(-3 \cdot 7^3 + 3 \cdot 19 q_4 q_5 - 3 \cdot 7 \cdot 167 + 2 \cdot 11 \cdot 251 - 2^3 \cdot 3^2 \cdot 53 \\ &\quad + 2^3 \cdot 3 \cdot 143 a_2^3 a_3), \\ f_{4,0} &= 12(-2^2 \cdot 11 \cdot 13 a_2 I_1 + 2 \cdot 3^2 a_3 q_5 + 2^3 \cdot 3^2 \cdot 5 q_4^2 + 2^3 \cdot 5 \cdot 59 a_2^2 q_4 \\ &\quad - 2^2 \cdot 3 \cdot 143 a_2 a_3^2), \\ f_{3,1} &= 12(2^2 \cdot 157 + 2 \cdot 3^2 \cdot 5 \cdot 41 - 2 \cdot 3^2 \cdot 5 - 2 \cdot 1039 - 2^2 \cdot 3 \cdot 143), \\ f_{2,2} &= 12(619 - 3^2 \cdot 829 - 2^4 \cdot 3 - 2 \cdot 667 + 2^2 \cdot 3 \cdot 461 - 2 \cdot 3 \cdot 143 a_2^4), \\ f_{5,0} &= 36(3 \cdot 5^2 a_2 q_5 + 3 \cdot 7 \cdot 11 a_3 q_4), \\ f_{4,1} &= 36(-1093 + 3^2 \cdot 47 + 2^3 \cdot 143 a_2^2 a_3), \\ f_{3,2} &= 36(7 \cdot 389 - 3 \cdot 11 \cdot 17 - 2^2 \cdot 3 \cdot 53), \\ f_{2,3} &= 36(5^2 \cdot 101 - 3^2 \cdot 79 - 2^2 \cdot 3 \cdot 53), \\ f_{5,1} &= 12(2 \cdot 3 \cdot 197 I_1 - 2^3 \cdot 7^2 \cdot 11 a_2 q_4 + 2^3 \cdot 3^2 \cdot 53 a_3^2), \\ f_{4,2} &= 12(2 \cdot 3 \cdot 197 - 2^2 \cdot 5 \cdot 137 + 2^3 \cdot 3^2 \cdot 53 - 2^2 \cdot 3 \cdot 143 a_2^3), \\ f_{3,3} &= 12(2^3 \cdot 3 \cdot 53 - 2^2 \cdot 19 \cdot 31 - 2^3 \cdot 3^2 \cdot 53 + 2^2 \cdot 3 \cdot 143), \\ f_{2,4} &= 12(2^4 \cdot 3 \cdot 5^2 - 2^2 \cdot 5^2 \cdot 31 + 2^3 \cdot 3^2 \cdot 53 - 2^2 \cdot 3 \cdot 143), \\ f_{5,2} &= 2^4 \cdot 3^3 \cdot 317 q_5 - 2^6 \cdot 3^2 \cdot 151 a_2 a_3, \quad f_{4,3} = 2^4 \cdot 3^3 \cdot 307 q_5 - 2^9 \cdot 3^2 a_2 a_3, \\ f_{3,4} &= 2^4 \cdot 3^5 \cdot 5 \cdot 7 q_5 - 2^9 \cdot 3^2 a_2 a_3, \quad f_{2,5} = 2^3 \cdot 3^3 \cdot 619 q_5 - 2^6 \cdot 3^2 \cdot 151 a_2 a_3, \\ f_{5,3} &= 2^6 \cdot 3^2 \cdot 11 q_4 + 2^5 \cdot 3^2 \cdot 143 a_2^2, \quad f_{4,4} = -2^6 \cdot 3^2 \cdot 13 q_4 - 2^5 \cdot 3^2 \cdot 143 a_2^2, \\ f_{3,5} &= 2^5 \cdot 3^2 \cdot 7 q_4 + 2^5 \cdot 3^2 \cdot 143 a_2^2, \quad f_{5,4} = f_{4,5} = 2^6 \cdot 3^3 \cdot 53 a_3, \\ f_{5,5} &= 2^6 \cdot 3^2 \cdot 143 a_2. \end{aligned} \right\} \quad (34)$$

Thus the coefficient of ξ_1 in the collineation can be expressed in terms of x_1, K_2, K_3, K_4, K_5 linearly with coefficients which are functions of a_2, a_3, q_4, q_5, I_1 and b_1, b_1^2, \dots, b_1^5 .

As for the second invariant the sum $K_{1,1} + \dots + K_{6,6}$ if not identically zero would furnish the self-apolarity invariant of the collineation which we could define to be I_2 .

The result of operating with C on Q is a $C_{19,0,1}$, i. e., the second linear covariant L_2 . Hence,

$$5/2 L_2 = \sum_{(1)}^6 (6C_1 S_1 + \sum_{(2)}^5 C_2 S_{12}) x_1. \quad (35)$$

The collineation K sends L_1 into the third linear covariant L_3 . This likewise is sent into the fourth linear covariant L_4 by the same collineation.

The three remaining invariants I_3, I_4, I_5 are obtained by operating with C on L_1, L_2, L_3 , respectively.

The Lines of Electric Force Due to a Moving Electron.

BY FRANCIS D. MURNAGHAN.

INTRODUCTION.

The values of the electric and magnetic forces in the electromagnetic field due to a moving point charge, or electron, have been given by Liénard,* Wiechert,† Heaviside,‡ and, in detail, by Abraham.§ The expressions are, however, so complicated in the case of perfectly general motion that the form of the lines of force was not obtained except for the simplest case—that of uniform motion in a straight line. The field of force for this case was known|| long before that for the general case was worked out, and it was shown that the lines of electric force at any time t were straight lines through the position of the electron at that time t ; a result which may be expressed by saying that an electron in uniform motion along a straight line convects its field along with it.

The form of the lines of electric force is of interest in connection with a theory, advanced by Sir J. J. Thomson,¶ of the structure of the ether and the nature of Röntgen rays and light. Following Faraday,** Thomson regards the lines of force not merely as an abstract mathematical concept but as physical realities. He makes, however, one important modification in Faraday's theory. If we have two charges of opposite sign they will be joined by lines of force, the direction of a line at any point being the direction of the electric force at that point. These are the lines of force which Faraday imagined as physical realities endowed with such properties as tension along their length and mutual transverse repulsion. In Thomson's modification of the theory each charge is supposed to carry its own lines of force with it independently of the presence of other charges. In a field containing many charges there will be many lines

* *L'éclairage électrique*, Vol. XVI (1898), pp. 5, 53, 106.

† *Arch. néerlandaises* (2), Vol. V (1900), p. 549.

‡ *Nature*, Vol. LXVII (1902), p. 6.

§ *Ann. d. Phys.*, Vol. XIV (1904), p. 236; "Theorie der Elektrizität," Band 2, §§ 13–15, Leipzig (1914).

|| O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

¶ Thomson, *Proc. Camb. Phil. Soc.*, Vol. XV (1909), p. 65.

** A full account of Faraday's theory is given by Thomson in his "Recent Researches in Electricity and Magnetism," Chapter I.

of force crossing at any point and these give a resultant electric force whose direction is that of the Faraday line at the point. Accordingly, the lines of force attached to a point charge are conceived by Thomson as more fundamental than the idea of the point charge itself and may, in fact, be used to define it.

To account for results obtained in experiments on the ionization of gases by Röntgen rays and ultra-violet light Thomson* has modified his theory by supposing that the lines of force are not distributed in a continuous manner round the point charge. (He had previously† made this modification in Faraday's concept.)

SECTION 1.

Recently a method was given by Bateman‡ which reduces the problem of finding the equations, at any time t , of the lines of electric force due to a point charge moving in any way, to the solution of a differential equation of Riccati's type. It will be shown that this equation can always be integrated if the path of the electron lies wholly in a plane. The following cases, which are those of most interest, of motion in a plane have been worked out in detail:

- (a) Rectilinear motion with any velocity and acceleration;
- (b) Uniform motion in a circle.

It has been found possible to integrate the equation in one case of motion in three dimensions—that of uniform motion in a circular helix. Further, it has been found that if one solution of the Riccatian equation is known a second can be derived from it and accordingly the general solution is reduced to the evaluation of a single quadrature.

It will be convenient to give here a brief account of the method by which Bateman obtained the Riccatian equation referred to. Let C be the curve along which the electron is moving and let $x=\xi$, $y=\eta$, $z=\zeta$ be the coordinates of the point of C , occupied by the electron at time τ , so that ξ , η , ζ are functions of τ . Denote differentiations with regard to τ by primes so that the velocity of the electron is $\bar{v} = (\xi', \eta', \zeta')$ and its acceleration is $\bar{v}' = (\xi'', \eta'', \zeta'')$; where bars are used to distinguish vector quantities from scalars. Let P be any point in space. Then if x, y, z are the coordinates of P , a disturbance emanating from the electron at time τ will reach P at time t if $c(t-\tau)=r$ where c is the velocity of light and $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. Denote by \bar{r} the vector

* *Phil. Mag.*, Vol. XIX (1910), p. 301.

† *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 417. (For this and other theories as to the structure of the ether see Bateman "Electrical and Optical Wave Motion," Camb., 1915, Chapter 8.)

‡ *Bulletin Amer. Math. Soc.*, 2d Series, Vol. XXI (1915), No. 6, p. 308; *American Journal*, April (1915).

OP (where O is the position of the electron at time τ) and let (l, m, n) be the direction cosines of \bar{r} . Then the electric force at P at time t is known* to be

$$E = \frac{e}{r^2} \left\{ \frac{\bar{r}}{r} - \frac{\bar{v}}{c} \right\} \left\{ 1 - \frac{v^2}{c^2} + \frac{(\bar{v}' \bar{r})}{c^2} \right\} \left(\frac{\partial \tau}{\partial t} \right)^3 - \frac{e \bar{v}'}{r c^2} \left(\frac{\partial \tau}{\partial t} \right)^2,$$

where e is the charge on the electron, round brackets denoting scalar products.

If we differentiate the equation $r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$ we get $\frac{\partial r}{\partial \tau} = \lambda - c$ where $\lambda = c - \frac{1}{r}(\bar{r} \bar{v})$ and since $c(t - \tau) = r$ we have $c \frac{\partial t}{\partial \tau} = \lambda$. We shall be interested in the direction only of E at P and an easy transformation of the expression for E shows that it has the direction of the vector

$$(c^2 - v^2) \left\{ \bar{v} - c \frac{\bar{r}}{r} \right\} + r \left\{ \lambda \bar{v}' + \left(\bar{v} - c \frac{\bar{r}}{r} \right) p \right\},$$

where $p = l\xi'' + m\eta'' + n\zeta'' = \frac{1}{r}(\bar{r} \bar{v}')$.

Let P' be a point consecutive to P and let its coordinates be $x + dx, y + dy, z + dz$. P' will be reached at time t by a disturbance which has emanated from the electron at time $\tau + d\tau$; at this time let O' be the position of the electron so that $\xi + d\xi, \eta + d\eta, \zeta + d\zeta$ are the coordinates of O' . The x component of the vector equation $\overline{PP'} = \overline{PO} + \overline{OO'} + \overline{O'P'}$ may be written

$$dx = -rl + d\xi + c(l + dl)(t - \tau - d\tau) \quad \text{or} \quad \frac{dx}{d\tau} = r \frac{dl}{d\tau} + \frac{d\xi}{d\tau} - cl.$$

Thus, if \bar{s} is the vector $\overline{PP'}$ we have $\bar{s}' = \left(\bar{v} - c \frac{\bar{r}}{r} \right) + r(l', m', n')$ where, as previously, (l', m', n') is used to denote the vector whose components are l', m', n' . Comparing this with the expression giving the direction of the electric force at P , we see that PP' is the direction of the electric force at P if the three equations

$$(c^2 - v^2) \frac{d}{d\tau} (l, m, n) = \lambda (\xi'', \eta'', \zeta'') + p (\xi' - cl, \eta' - cm, \zeta' - cn) \dots \quad (A)$$

are satisfied. From the manner in which these equations were derived we see that, if (l, m, n) is a solution of these equations satisfying also the condition $l^2 + m^2 + n^2 = 1$; the equations

$$x = \xi + c(t - \tau)l, \quad y = \eta + c(t - \tau)m, \quad z = \zeta + c(t - \tau)n$$

(where τ is a parameter satisfying the inequality $\tau < t$) give a line of electric force at time t . It will be noticed that, if at each position of the electron we imagine a particle projected with velocity c in the direction (l, m, n) , the

* Abraham, "Theorie der Elektrizität."

aggregate of these particles will at any time t form a line of electric force. All the lines of force at any time t pass through the position of the electron at that time.

In order to obtain the solutions (l, m, n) of (A) which satisfy $l^2 + m^2 + n^2 = 1$ put $\sigma = \frac{l+im}{1+n} = \frac{1-n}{l-im}$ where $i^2 = -1$. After an easy reduction we obtain the Riccattian equation

$$2(c^2 - v^2) \frac{d\sigma}{d\tau} = \phi''(c - \zeta') + \phi' \zeta'' - 2\sigma \{ c \zeta'' + i(\xi' \eta'' - \xi'' \eta') \} + \sigma^2 \{ \psi' \zeta'' - (c + \zeta') \psi'' \}$$

where $\phi = \xi + i\eta$ and ψ is its conjugate $\xi - i\eta$.

If σ is a solution of this equation and if its conjugate is s , $l = \frac{\sigma + s}{1 + \sigma s}$, $m = -i \frac{\sigma - s}{1 + \sigma s}$, $n = \frac{1 - \sigma s}{1 + \sigma s}$ is a solution of the equations (A) which satisfies $l^2 + m^2 + n^2 = 1$.

It is interesting to note that the equations (A) are a generalization of the equations

$$\frac{d}{d\tau} (l, m, n) = (mr - nq, np - lr, lq - mp) \dots \quad (B)$$

to which they reduce on making $c=0$ and $v^2(p, q, r) = [\bar{v}' \bar{v}]$ the square brackets being used to denote a vector product. These equations occur in the kinematics of a rigid body and have been considered at length by Darboux.* Several writers† have extended them to the case of four or more variables and have obtained results similar to those found by Darboux.

§ 2. ASSOCIATED DIRECTIONS OF PROJECTION.

A solution (l, m, n) of the equations

$$(c^2 - v^2) \frac{d}{d\tau} (l, m, n) = \lambda(\xi'', \eta'', \zeta'') + p(\xi' - cl, \eta' - cm, \zeta' - cn),$$

which have been mentioned in the preceding paragraph, will be called a *direction of projection* provided $l^2 + m^2 + n^2 = 1$. It will now be shown that to every direction of projection there corresponds, in an involutory manner, a second direction which shall be called the *associated direction of projection*.

Let O be the position at time τ of the moving electron so that the Cartesian coordinates of O are $\xi(\tau), \eta(\tau), \zeta(\tau)$. Let OP be a direction of projection at O so that (l, m, n) , the direction cosines of OP , satisfy the fundamental

* "Théorie des Surfaces," Tome 1, Chapters 2 and 3.

† Craig, Cole, Eiesland, *Amer. Journal*, Vol. XX (1898); Hatzidakis, *Amer. Journal*, Vol. XXIII (1901); Eiesland, *Amer. Journal*, Vol. XXVIII (1906); Matsumoto, *Memoirs of Coll. of Science*, Kyoto Imp. Univ., Vol. I, No. 7, January (1916).

equations (A) of § 1. Let Q be the position which would have been occupied by the electron at a time $t > \tau$ if it had continued to move with velocity $v = (\xi', \eta', \zeta')$ along the tangent to its path at O . Thus, the coordinates of Q are

$$(x_1, y_1, z_1) = (\xi + \xi' \overline{t - \tau}, \eta + \eta' \overline{t - \tau}, \zeta + \zeta' \overline{t - \tau}).$$

If now we make $OP = c(t - \tau)$ and draw a sphere with centre O and radius OP , then PQ will meet the sphere again in a point P' . OP' is the associated direction of projection at O . To prove this write $c(t - \tau) = r$ so that the coordinates of P are $(x_0, y_0, z_0) = (\xi + rl, \eta + rm, \zeta + rn)$ and PQ has the equations $x - x_1 / x_1 - x_0 = y - y_1 / y_1 - y_0 = z - z_1 / z_1 - z_0 = \theta$, say. But $x_1 - x_0 = (t - \tau)(\xi' - cl)$ and so we have three equations of the type $x - \xi = \xi'(t - \tau) + \theta(\xi' - cl)(t - \tau)$. Let (x, y, z) be the coordinates of P' . P' being on PQ we have $x - \xi = (t - \tau)\{\xi' + \theta_1(\xi' - cl)\}$ where θ_1 is the value of the parameter θ giving P' . Since P' is on the sphere we have the equation

$$(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = c^2(t - \tau)^2$$

and from these two equations we get

$$\theta_1^2 \{c^2 + v^2 - 2\frac{c}{r}(\bar{r}\bar{v})\} + 2\theta_1 \{v^2 - \frac{c}{r}(\bar{r}\bar{v})\} + v^2 - c^2 = 0.$$

The roots of this equation are $\theta_1 = -1$ and $\theta_1 = (c^2 - v^2) / [c^2 + v^2 - 2\frac{c}{r}(\bar{r}\bar{v})]$.

The root $\theta_1 = -1$ is the parameter of the point P itself. The second root is the parameter of P' and so if (L, M, N) are the direction cosines of OP' , so that $x - \xi = c(t - \tau)L$, etc., we obtain the following three equations for L, M, N :

$$(\xi' - cL, \eta' - cM, \zeta' - cN) = h(v^2 - c^2)(\xi' - cl, \eta' - cm, \zeta' - cn),$$

where $h^{-1} = c^2 - 2\frac{c}{r}(\bar{r}\bar{v}) + v^2$. If as in § 1 we use the abbreviations

$$\lambda = c - \frac{1}{r}(\bar{r}\bar{v}), \mu = c^2 - v^2, \text{ then } h^{-1} = 2c\lambda - \mu.$$

Write \bar{R} to denote the vector OP' , and Γ to denote the expression $c - \frac{1}{r}(\bar{R}\bar{v})$; then from the values obtained for (L, M, N) it is easy to deduce the equation $\Gamma = h\lambda\mu$.

In order to prove that OP' is a direction of projection we have merely to show that (L, M, N) satisfy the equations (A) of § 1. In other words, it must be shown that the three equations

$$\mu \frac{d}{d\tau}(L, M, N) = \Gamma(\xi'', \eta'', \zeta'') + P(\xi' - cL, \eta' - cM, \zeta' - cN)$$

are true, where $P = L\xi'' + M\eta'' + N\zeta'' = \frac{1}{r}(\bar{R}\bar{v})$ and, as before, \bar{v} is the vector acceleration of the electron at O . If now we multiply the equations (A) by ξ', η', ζ' , respectively, and add they give $\mu(p+q) = \lambda\{cp + (\bar{v}\bar{v}')\}$ where $q = l'\xi' + m'\eta' + n'\zeta'$. This equation is equivalent to

$$-\mu\lambda' = \lambda\{cp + (\bar{v}\bar{v}')\}.$$

Again $\xi' - cL = -\mu h(\xi' - cl)$. Our object is to obtain L' so we differentiate this equation and it gives

$$cL' - \xi'' = \mu h(\xi'' - cl') - 2h(\bar{v}\bar{v}')(\xi' - cl) - 2\mu h^2(\xi' - cl)[c\lambda' + (\bar{v}\bar{v}')].$$

On combining the first two terms on the right-hand side of this equation with the term $-\xi''$ on the left-hand side we obtain, after substituting for l' its value from the equation (A) of § 1 and for λ' its value just given

$$cL' = h\{c\lambda\xi'' - (\xi' - cl)[cp + 2(\bar{v}\bar{v}')]\} + 2h^2(\xi' - cl)\{c^2\lambda p + (c\lambda - \mu)(\bar{v}\bar{v}')\}$$

$$\text{or} \quad L' = h\{\lambda\xi' - (\xi' - cl)p\} + 2h^2\lambda(\xi' - cl)\{cp - (\bar{v}\bar{v}')\}.$$

Again from the values already given for (L, M, N) we derive

$$cP = (\bar{v}\bar{v}') + \mu h\{(\bar{v}\bar{v}') - cp\} \text{ or } P = h[2\lambda(\bar{v}\bar{v}') - \mu p] \text{ since } h^{-1} = 2c\lambda - \mu.$$

Substituting these values of L', P in the equation $\mu L' = \Gamma\xi'' + (\xi' - cL)P$ we see on dividing across by $\mu h p(\xi' - cl)$ that this equation is true if $1 - 2hc\lambda + \mu h = 0$ which is so since $h^{-1} = 2c\lambda - \mu$.

This proves the theorem stated. If we know a particular solution $OP = (l, m, n)$ of the equations (A) we can obtain a second solution $OP' = (L, M, N)$ in the manner described at the beginning of this paragraph. Darboux* has given a similar result for the equations (B) of § 1.

There is one particular case, when the associated direction coincides with the original direction of projection, in which we can not obtain by this method a new solution of the differential equations. In this case PP' is a tangent line to the sphere and so $l\xi' + m\eta' + n\zeta' = v\frac{c}{v}$ or $\lambda = 0$. It is easy, as a matter of fact, to verify that if we put $\lambda = 0$ in the equations for L, M, N that $L = l, M = m, N = n$. In this case the velocity v of the electron is greater than c , the velocity of light. From the equation for λ' we see that when $\lambda = 0$ λ' also $= 0$. In the cases of motion for which the solution has been worked out it will be seen that there are always a single infinity of solutions compatible with $\lambda = 0$.

* "Théorie des Surfaces," Tome I (1887), p. 23.

§ 3. GENERAL SOLUTION WHEN MOTION IS IN ONE PLANE.

Let θ be the angle between OP and OQ so that $v \cos \theta = l\xi' + m\eta' + n\zeta'$. It will now be seen that if the associated directions OP, OP' are distinct and equally inclined to OQ , the motion of the electron is in one plane and the solution can be found. OP and OP' are equally inclined to OQ if $\cos \theta = \frac{v}{c}$ for then PP' is perpendicular to OQ . Also since $\lambda = c - v \cos \theta$ we have $\mu = c\lambda$, and so $-c\lambda' = 2(\bar{v}\bar{v}')$. On substitution in the equation $-\mu\lambda' = \lambda\{cp + (\bar{v}\bar{v}')\}$ of § 2 we get $-c\lambda' = cp + (\bar{v}\bar{v}')$ or $cp = (\bar{v}\bar{v}')$. Substituting in the first of the equations (A) this gives $\mu(\xi'' - cl') = -(\xi' - cl)(\bar{v}\bar{v}')$ which is at once integrable, giving $\xi' - cl = a(c^2 - v^2)^{\frac{1}{2}}$ where a is a constant of integration. Similarly, we obtain the equations $\eta' - cm = b(c^2 - v^2)^{\frac{1}{2}}$, $\zeta' - cn = d(c^2 - v^2)^{\frac{1}{2}}$ where b and d are constants of integration. On multiplication by ξ', η', ζ' , respectively, and addition, these three integrals give $a\xi' + b\eta' + d\zeta' = 0$ (remembering $v^2 = c^2 - c\lambda$).

Similarly, on using ξ'', η'', ζ'' as multipliers, we get, since $(\bar{v}\bar{v}') = cp$,

$$a\xi'' + b\eta'' + d\zeta'' = 0.$$

Thus, the velocity and the acceleration of the electron are in the same plane; hence, if a solution of the fundamental equations can be found which satisfies the equation $c \cos \theta = v$, the motion of the electron is in one plane. Conversely, if the motion is plane it is easy to pick out a solution for which $c \cos \theta = v$. The theorem of associated directions, then, gives a second solution which also satisfies $c \cos \theta = v$. Knowing these two solutions the integration of the Riccatian equation of § 1 is effected by means of a single quadrature.

Take the plane of motion to be $z=0$ so that $\zeta'=0, \zeta''=0$. It is evident that $cl=\xi', cm=\eta'$ satisfy the first two equations. These values of l, m satisfy $c \cos \theta = v$. To determine n we have the condition $l^2 + m^2 + n^2 = 1$, giving $cn = \pm (c^2 - v^2)^{\frac{1}{2}}$. Either of these values of n satisfies the third equation which is, in this case, $\mu n' = -cn(l\xi'' + m\eta'')$. Hence, the solutions which give $c \cos \theta = v$ are $cl=\xi', cm=\eta', cn=\pm\mu^{\frac{1}{2}}$.

To obtain the general direction of projection it is necessary to solve the Riccatian equation. In the case of plane motion this is somewhat simplified and is $2(c^2 - v^2) \frac{d\sigma}{d\tau} = c\phi'' - 2i\sigma(\xi'\eta'' - \xi''\eta') - c\sigma^2\psi''$ where $\sigma(1+n) = l+im$.

Substituting for l, m, n the values just given we have the two particular solutions σ_1, σ_2 given by $c(1+\omega)\sigma_1 = \phi', c(1-\omega)\sigma_2 = \phi'$ where $c\omega = (c^2 - v^2)^{\frac{1}{2}}$. It is easy to verify that σ_1, σ_2 satisfy the Riccatian equation.

The general solution of the Riccatian equation is, therefore,*

$$\sigma = \sigma_2 + \frac{\sigma_1 - \sigma_2}{1 + A e^{(\sigma_1 - \sigma_2) R d\tau}},$$

where A is a constant of integration and $2(c^2 - v^2)R = -c\psi''$. A is, in general, complex since σ is complex.

Thus, when the motion is plane, the problem is reduced to the integration

$$\int (\sigma_1 - \sigma_2) R d\tau \text{ or } c \int \frac{(\xi'' - i\eta'') d\tau}{(\xi' - i\eta') \sqrt{c^2 - \xi'^2 - \eta'^2}}.$$

The integration can always be effected if the scalar quantity v is constant, and in this case the integral is $[c/(c^2 - v^2)^{1/2}] \log \psi'$, so that

$$\sigma = \sigma_2 + \frac{\sigma_1 - \sigma_2}{1 + A \psi'^{\frac{1}{v}}}.$$

If $\eta = 0$, $\eta' = 0$, $\eta'' = 0$, it is evident that the integration can be carried out. This is the case of rectilinear motion which will be treated in detail. The remaining case of most interest is that of uniform motion in a circle and the solution in this case given in terms of the trigonometric functions. In general, the explicit determination of the equations of the lines of force depends on the evaluation of the integral given above.

§ 4. PARTICULAR CASES OF PLANE MOTION.

(a) Rectilinear Motion.

Taking the x -axis as the line of motion we see from § 3 that the solution depends on the integration of $c \int \frac{\xi'' d\tau}{\xi' \sqrt{c^2 - \xi'^2}}$. In this particular case, however, the general solution can be obtained at once directly from the Riccatian equation which becomes $2(c^2 - \xi'^2)\sigma' = c\xi''(1 - \sigma^2)$.

This gives $\log \frac{1 + \sigma}{1 - \sigma} = \frac{1}{2} \log \frac{c + v}{c - v} + \log A$ (for $\xi' = v$). Write $A = \alpha + i\beta$

and we have $\frac{1 + \sigma}{1 - \sigma} = (\alpha + i\beta) \left(\frac{c + v}{c - v} \right)^{1/2}$. Whence,

$$\sigma = \{ (\alpha + i\beta) (c^2 - v^2)^{1/2} - (c - v) \} \div \{ (\alpha + i\beta) (c^2 - v^2)^{1/2} + (c - v) \}.$$

From this point we must distinguish between the cases $v \lesseqgtr c$.

Linear Motion When $v < c$.

$(c^2 - v^2)^{1/2}$ is real and so if s denotes the conjugate of σ we have

$$s \{ (\alpha - i\beta) (c^2 - v^2)^{1/2} + (c - v) \} = (\alpha - i\beta) (c^2 - v^2)^{1/2} - (c - v).$$

* Cohen, "Differential Equations," p. 175.

Forming the product σs we find

$$\begin{aligned}\sigma s \{ (\alpha^2 + \beta^2) (c^2 - v^2) + (c - v)^2 + 2(c - v)(c^2 - v^2)^{\frac{1}{2}} \alpha \} \\ = (\alpha^2 + \beta^2) (c^2 - v^2) + (c - v)^2 - 2(c - v)(c^2 - v^2)^{\frac{1}{2}} \alpha,\end{aligned}$$

and since $(1 + \sigma s)l = \sigma + s$, $(1 + \sigma s)im = \sigma - s$, $(1 + \sigma s)n = 1 - \sigma s$ we have

$$Rl = (c + v)(\alpha^2 + \beta^2) - (c - v), \quad Rm = 2\beta(c^2 - v^2)^{\frac{1}{2}}, \quad Rn = 2\alpha(c^2 - v^2)^{\frac{1}{2}},$$

where $R = (c + v)(\alpha^2 + \beta^2) + (c - v)$.

The lines of force at time t are, with these values for l, m, n ,

$$x = \xi + c(t - \tau)l, \quad y = c(t - \tau)m, \quad z = c(t - \tau)n,$$

and α, β are two arbitrary real constants.

It is seen that $\alpha y = \beta z$ or the lines of force lie in planes through the direction of motion a result evident, à priori, from symmetry. If v is constant, so also are l, m, n , and we get the well-known result* that in the case of uniform motion in a right line the line of electric force at time t through any point x, y, z is got by joining x, y, z to the position of the electron at that time t . Since the lines of force are symmetrical round the direction of motion it is sufficient to consider the lines in the plane $z = 0$, so that $\alpha = 0$. For instance, in the case of simple harmonic motion we may write $\xi = a \cos p\tau$ and the lines of electric force in the xy -plane are $x = a \cos p\tau + c(t - \tau)l$, $y = c(t - \tau)m$, where

$$\begin{aligned}l \{ \beta^2 (c - ap \sin p\tau) + (c + ap \sin p\tau) \} &= \beta^2 (c - ap \sin p\tau) - (c + ap \sin p\tau), \\ m \{ \beta^2 (c - ap \sin p\tau) + (c + ap \sin p\tau) \} &= 2\beta (c^2 - a^2 p^2 \sin^2 p\tau)^{\frac{1}{2}}.\end{aligned}$$

The single arbitrary constant β gives the single infinity of lines in the plane. To get an idea of the shape of the lines take the particular line $\beta = 1$. Then, $cl = -ap \sin p\tau$, $cm = (c^2 - a^2 p^2 \sin^2 p\tau)^{\frac{1}{2}}$; put $t = 0$ and $\tau = -\tau'$ so that τ' takes positive values. Then, $x = a \cos p\tau' + c\tau'l$, $y = c\tau'm$, where $cl = ap \sin p\tau'$. Thus, $x = a(\cos p\tau' + p\tau' \sin p\tau')$, $y = \tau'(c^2 - a^2 p^2 \sin^2 p\tau')^{\frac{1}{2}}$, where τ' takes positive values. If a is small we have at distances far from the origin

$$x^2 + y^2 = c^2 \tau'^2, \quad x = a(\cos p\tau' + p\tau' \sin p\tau').$$

The line of force is of an oscillatory character with increasing amplitude about the line $x = 0$.

Case of Uniformly Accelerated Motion in a Straight Line.

Let g be the uniform acceleration and suppose the electron to start from rest at the origin at time $\tau = 0$. Then, $\xi = \frac{1}{2}g\tau^2$, $v = g\tau$ so that $Rl = \beta^2(c + g\tau) - (c - g\tau)$, $Rm = 2\beta(c^2 - g^2\tau^2)^{\frac{1}{2}}$, where $R = \beta^2(c + g\tau) + (c - g\tau)$. Consider the particular line $\beta = 1$. Then, $R = 2c$ and $cl = g\tau$ and the equations to the line of force are $x = \frac{1}{2}g\tau^2 + (t - \tau)g\tau$, $y = (t - \tau)(c^2 - g^2\tau^2)^{\frac{1}{2}}$.

* J. J. Thomson, *Phil. Mag.*, Vol. XI (1881), p. 229; O. Heaviside, *Phil. Mag.*, Vol. XXVII (1889), p. 324.

These equations take a somewhat simpler form if we transfer the origin to the position of the electron at time t . Then, $x = -\frac{1}{2}g(t-\tau)^2$ and taking $t-\tau$ as the parameter s we have $x = -\frac{1}{2}gs^2$, $y^2 = s^2\{c-g(t-s)\}\{c+g(t-s)\}$, where s takes values from t to zero. On eliminating s this is a quartic curve

$$\left\{y^2 + 4x^2 + \frac{2x}{g}(c^2 - g^2t^2)\right\}^2 + 8gt^2x^3 = 0.$$

It is easy to see how the lines of force due to a point charge in rectilinear motion are altered in shape when the moving charge is accelerated or retarded. For example, imagine the charge to have been moving with uniform velocity for a long time and, then, to be suddenly affected with a retardation g , and consider, as above, the line $\beta=1$ for which $l=\frac{v}{c}$; the electron, being at $\tau=0$

at $x=0$, is reduced to rest at time $\tau=\frac{v}{g}$. Consider the line of force at this

time. The part given by the aggregate of particles imagined projected at times $\tau < 0$ is a straight line which would pass if produced through the point

$x=\frac{v^2}{g}$. At the point where this straight line meets the line $cl=v$ through the

origin, the line of force bends and passes through the point $x=\frac{v^2}{2g}$ in a direction

perpendicular to the line of motion. At any later time t the line of force is made up of three parts: (1) A straight line part which goes, if produced,

through $x=vt$, (2) a bent part beginning where this line meets the sphere

$x^2+y^2+z^2=c^2t^2$ and ending at $x=\frac{v^2}{2g}$, $y=c\left(t-\frac{v}{g}\right)$, and (3) the part of the line

$x=\frac{v^2}{2g}$ from this point to $y=0$. A similar result holds for the lines got from

other values of β . It is evident that if the electron is stopped instantaneously we can get the lines of force in the following way: Draw the pencil of lines through $x=vt$, $y=0$. The sphere with centre at origin and radius ct cuts these lines in points where the lines of force bend suddenly, the remaining portions being the radii of this sphere. This explains how the discontinuity due to the sudden stoppage of an electron spreads out in a spherical wave. The commonly accepted theory of Röntgen rays is that they are pulses of this type due to the sudden stoppage of cathode particles. It is important to notice that in all cases the direction of projection is determined by the velocity at the point of projection and is independent of its acceleration.

If we give particular values to v, g, c , it is easy to draw the bent portion of the line of force due to the retarded part of the motion. For instance, take $v=4, g=c=8$; then the electron is reduced to rest in time $\tau = \frac{1}{2}$, its position at that time being $x=1$. Taking $\beta=1$ we have $l = \frac{v}{c}$. First take $t = \frac{1}{2}$, so that we are drawing the line for the instant when the electron is reduced to rest. Then, if $0 < \tau < \frac{1}{2}$, $l = \frac{4-8\tau}{8} = \frac{1}{2} - \tau$, $\xi = 4(\tau - \tau^2)$. The distance from ξ to the corresponding point on the line of force is $c(t - \tau) = 4(1 - 2\tau) = 8l$. It is found that the bent portion of the line of force resembles closely the part in the positive quadrant of an ellipse whose axes are in the ratio of 10:34. Having drawn the line of force at time $t = \frac{1}{2}$ we get the form of the same line at any subsequent time by producing the radii vectores from ξ to the corresponding point on the line of force at time $t = \frac{1}{2}$, a constant distance. This method of deducing the successive positions of a given line of force from the position at any one time follows from the equations of the lines of force and is applicable in the general case.

Motion in a Straight Line When $v > c$.

In the equation

$$\{(\alpha + i\beta)(c^2 - v^2)^{\frac{1}{2}} + (c - v)\}\sigma = \{(\alpha + i\beta)(c^2 - v^2)^{\frac{1}{2}} - (c - v)\}$$

$(c^2 - v^2)^{\frac{1}{2}}$ is now a pure imaginary. Writing it in form $i(v^2 - c^2)^{\frac{1}{2}}$ we see that s , the conjugate of σ , is given by the equation

$$\{(\alpha - i\beta)(v^2 - c^2)^{\frac{1}{2}} + i(c - v)\}s = (\alpha - i\beta)(v^2 - c^2)^{\frac{1}{2}} - i(c - v).$$

From these we obtain as before

$$Rl = (\alpha^2 + \beta^2)(v + c) - (v - c), \quad Rm = 2\alpha(v^2 - c^2)^{\frac{1}{2}}, \quad Rn = 2\beta(v^2 - c^2)^{\frac{1}{2}},$$

where $R = (\alpha^2 + \beta^2)(v + c) + (v - c)$.

The lines of force lie in planes through the axis of x ; putting $\beta=0$ we obtain the lines in the plane xy . It will be noticed that, for these, if $\alpha=1$, $l = \frac{c}{v}$, and so $\lambda=0$. Hence, in the case of rectilinear motion with constant velocity $v > c$ the generators of the cone $l = \frac{c}{v}$ are directions of projection. The cone with vertex at $x = \xi(t)$ and semi-vertical angle $\alpha = \sin^{-1} \frac{c}{v}$ accordingly

separates the regions where there are lines of electric force from those regions, where there are no lines of force. The generators of the cone are lines of electric force. This is a known result.* If v is not constant, these successive cones envelope a surface of revolution, the meridians on which are lines of electric force, and which enclose all the lines of force.

Motion in a Circle with Uniform Velocity.

Let p be the constant angular velocity and d the radius of the circle. Then, the plane of the circle being $z=0$, and the centre of the circle being the origin of coordinates, we have $\xi=a \cos p\tau$, $\eta=a \sin p\tau$, $\zeta=0$; whence, $\phi=ae^{ip\tau}$ and the particular solutions of the Riccatian equation are σ_1, σ_2 , where

$$c(1+\omega)\sigma_1=iap e^{ip\tau}, \quad c(1-\omega)=iap e^{ip\tau} \quad \text{and} \quad c^2\omega^2=c^2-v^2=c^2-a^2p^2.$$

On substituting these values in the general solution for plane motion it is found that

$$\sigma = ie^{ip\tau} \left[\frac{c(1+\omega)}{ap} + \frac{1}{(\alpha+i\beta)e^{\frac{-ip\tau}{\omega}} - ap/2c\omega} \right]^\dagger.$$

If $v < c$, ω is real; if $v > c$, ω is a pure imaginary, so there will be two distinct solutions according as $v \geq c$.

(a) *Case When $v < c$.*

Since ω is real, s is got by merely changing the sign of i wherever it appears in the expression for σ . Adding we have

$$\sigma + s = -\frac{2c(1+\omega)}{ap} \sin p\tau - 2 \frac{P \sin p\tau + Q \cos p\tau}{R},$$

where P, Q, R are defined by the equations

$$P = \alpha \cos p\tau/\omega + \beta \sin p\tau/\omega - ap/2c\omega,$$

$$Q = \alpha \sin p\tau/\omega - \beta \cos p\tau/\omega,$$

$$R = (\alpha^2 + \beta^2) + a^2 p^2 / 4c^2 \omega^2 - ap/c\omega (\alpha \cos p\tau/\omega + \beta \sin p\tau/\omega).$$

Similarly, on subtracting we find

$$-i(\sigma - s) = 2c(1+\omega)/ap \cos p\tau + 2 \frac{P \cos p\tau - Q \sin p\tau}{R},$$

and on multiplication

$$\sigma s = \frac{c^2(1+\omega)^2}{a^2 p^2} + \frac{2c(1+\omega)}{ap} \frac{P+1}{R};$$

* O. Heaviside, "Electrical Papers."

† The integral $\int \frac{\psi''}{\psi'} d\tau$ of § 3 is, in this case, $\frac{1}{\omega} \log \psi'$ and $\psi = ae^{-ip\tau}$.

from these equations we obtain $R(1+\sigma s)=S$, $R(1-\sigma s)=T$, where S and T are defined by the equations

$$S = (\alpha^2 + \beta^2) \frac{2}{1-\omega} + \frac{1-\omega}{2\omega^2} - \frac{2ap}{c\omega} \left(\alpha \cos \frac{p\tau}{\omega} + \beta \sin \frac{p\tau}{\omega} \right),$$

$$T = (\alpha^2 + \beta^2) \frac{2\omega}{\omega-1} + \frac{1-\omega}{2\omega},$$

and hence l, m, n are given by

$$l = \frac{\sigma+s}{1+\sigma s} = -\frac{ap}{c} \sin p\tau + \frac{2}{S} \left\{ \left(P + \frac{ap}{2c\omega} \right) \omega \sin p\tau - Q \cos p\tau \right\},$$

$$m = -i \frac{\sigma-s}{1+\sigma s} = \frac{ap}{c} \cos p\tau - \frac{2}{S} \left\{ \left(P + \frac{ap}{2c\omega} \right) \omega \cos p\tau + Q \sin p\tau \right\},$$

$$n = \frac{1-\sigma s}{1+\sigma s} = \frac{T}{S},$$

and with these values of l, m, n the parametric equations to the lines of force are $x=a \cos p\tau + c(t-\tau)l$, $y=a \sin p\tau + c(t-\tau)m$, $z=c(t-\tau)n$.

By a suitable choice of the constants α, β so as to get particular lines of force these expressions may be considerably simplified. Thus, if we put $\alpha=0$, $\beta=0$, we have $l=-(ap/c) \sin p\tau$, $m=(ap/c) \cos p\tau$, $n=\omega$, and the line of force is given by the equations

$$x=a \cos p\tau - ap(t-\tau) \sin p\tau, \quad y=a \sin p\tau + ap(t-\tau) \cos p\tau, \quad z=c\omega(t-\tau).$$

It is obvious that it lies on the hyperboloid of revolution

$$(x^2+y^2)/a^2 - p^2 z^2 / c^2 \omega^2 = 1$$

which contains the circle of motion. Writing this equation in the form

$$(x/a - pz/c\omega)(x/a + pz/c\omega) = (1+y/a)(1-y/a),$$

we see that the directions of projection are generators of the hyperboloid. The form of the line of force is evident if we consider its projection on the plane of motion. The equations of the projection are

$$x=a \cos p\tau - ap(t-\tau) \sin p\tau, \quad y=a \sin p\tau + ap(t-\tau) \cos p\tau, \quad z=0;$$

giving

$$(x-a \cos p\tau) + (y-a \sin p\tau) \tan p\tau = 0$$

$$\text{and } (x-a \cos p\tau)^2 + (y-a \sin p\tau)^2 = a^2 p^2 (t-\tau)^2.$$

These are the equations to an involute of the circle $x^2+y^2=a^2$; the involute meeting the circle at the position of the electron at time $\tau=t$. To obtain, then, a line of electric force due to an electron moving with velocity $v < c$ in a circle we unwrap a string which is wound round the circle, the tracing point at the end of the string leaving the circle at the position of the electron at time t . Having obtained this involute we project it by lines parallel to the axis of z on the hyperboloid $(x^2+y^2)/a^2 - p^2 z^2 / c^2 \omega^2 = 1$.

Lines of Force in the Plane of Motion.

To obtain these put $n=0$. Then, $T=0$ or $(\alpha^2 + \beta^2) = (1 - \omega/2\omega)^2$, and so we may write $2\omega\alpha = (1 - \omega) \cos \theta$, $2\omega\beta = (1 - \omega) \sin \theta$ and these give $\omega^2 S = (1 - \omega)k$ where $k = 1 - (ap/c) \cos(p\tau/\omega - \theta)$, and, finally,

$$kl = \sin p\tau \cos(p\tau/\omega - \theta) - (ap/c) \sin p\tau - \omega \cos p\tau \sin(p\tau/\omega - \theta),$$

$$km = -\cos p\tau \cos(p\tau/\omega - \theta) + (ap/c) \cos p\tau - \omega \sin p\tau \sin(p\tau/\omega - \theta).$$

The single constant θ gives the single infinity of lines in the plane of motion; from symmetry we need only consider a particular value of θ , say $\theta=0$; then,

$$kl = \sin p\tau \cos p\tau/\omega - (ap/c) \sin p\tau - \omega \cos p\tau \sin p\tau/\omega,$$

$$km = -\cos p\tau \cos p\tau/\omega + (ap/c) \cos p\tau - \omega \sin p\tau \sin p\tau/\omega,$$

where the Doppler factor k is now $1 - (ap/c) \cos p\tau/\omega$.

Circular Motion When $v > c$.

ω is now a pure imaginary and so it may be written $=i\theta$ where θ is real. The functions σ and s are given by the equations

$$\sigma = ie^{ip\tau} \left\{ \frac{c(1+i\theta)}{ap} + \frac{1}{(\alpha+i\beta)e^{-p\tau/\theta} + iap/2c\theta} \right\},$$

$$s = -ie^{-ip\tau} \left\{ \frac{c(1-i\theta)}{ap} + \frac{1}{(\alpha-i\beta)e^{-p\tau/\theta} - iap/2c\theta} \right\},$$

whence

$$D(\sigma + s) = -D \left(\frac{2c}{ap} \sin p\tau + \frac{2c\theta}{ap} \cos p\tau \right) + 2e^{-p\tau/\theta} \{ \beta \cos p\tau - \alpha \sin p\tau \} + \frac{ap}{c\theta} \cos p\tau,$$

where

$$D = \alpha^2 e^{-2p\tau/\theta} + \left\{ \beta e^{-p\tau/\theta} + \frac{ap}{2c\theta} \right\}^2.$$

Similarly,

$$-iD(\sigma - s) = \frac{2c}{ap} D(\cos p\tau - \theta \sin p\tau) + 2e^{-p\tau/\theta} \{ \alpha \cos p\tau + \beta \sin p\tau \} + \frac{ap}{c\theta} \sin p\tau$$

$$\text{and } \sigma s = 1 + \frac{2c}{apD} e^{-p\tau/\theta} (\alpha - \theta\beta).$$

Using these values we obtain, after some reduction, and on writing

$$F = (\alpha^2 + \beta^2) e^{-2p\tau/\theta} - a^2 p^2 / 4c^2 \theta^2,$$

$$G = (\alpha^2 + \beta^2) e^{-2p\tau/\theta} + a^2 p^2 / 4c^2 \theta^2 + \frac{c}{ap} e^{-p\tau/\theta} (\alpha + \beta/\theta),$$

the equations

$$l = -\frac{c}{ap} \left[\sin p\tau + \frac{\theta \{ F \cos p\tau + \frac{c}{ap} e^{-p\tau/\theta} (\theta\alpha + \beta) \sin p\tau \}}{G} \right],$$

$$m = \frac{c}{ap} \left[\cos p\tau - \frac{\theta \{ F \sin p\tau - \frac{c}{ap} e^{-p\tau/\theta} (\theta\alpha + \beta) \cos p\tau \}}{G} \right],$$

$$n = \frac{c}{ap} e^{-p\tau/\theta} \frac{(\theta\beta - \alpha)}{G}.$$

The lines in the plane of motion are obtained by writing $\alpha = \theta\beta$. Further, $\lambda=0$ reduces to $\theta\alpha + \beta=0$ and with this relation between the constants we find

$$\begin{aligned} l &= \left\{ \frac{c\theta}{ap} \cos p\tau \right\} \frac{1-x}{1+x} - \frac{c}{ap} \sin p\tau, \\ m &= \left\{ \frac{c\theta}{ap} \sin p\tau \right\} \frac{1-x}{1+x} + \frac{c}{ap} \sin p\tau, \\ n &= -\frac{4c\alpha\theta^2 e^{-p\tau/\theta}}{ap(1+x)}, \end{aligned}$$

where $x=4\alpha^2\theta^2 e^{-2p\tau/\theta}$. Thus there is a single infinity of lines of electric force whose directions of projection satisfy $\lambda=0$. Putting $\alpha=0$ we get that one of these which lies in the plane of motion and from symmetry this line is the same as all the lines in the plane of motion. Since for this line

$$l = -\frac{c}{ap} [\sin p\tau - \theta \cos p\tau], \quad m = \frac{c}{ap} [\cos p\tau + \theta \sin p\tau], \quad n = 0,$$

we have on writing $\sin \phi = \frac{c}{ap}$, $\cos \phi = \frac{c\theta}{ap}$ that $l = \cos(p\tau + \phi)$, $m = \sin(p\tau + \phi)$, and the equations to the line of force are

$$\begin{aligned} x &= a \cos p\tau + c(t-\tau) \cos(p\tau + \phi), \\ y &= a \sin p\tau + c(t-\tau) \sin(p\tau + \phi). \end{aligned}$$

The direction of projection makes a fixed angle with the radius of the circle to the point of projection; the line, therefore, winds round the circle in a manner very similar to its involutes.

Motion in a Helix with Uniform Velocity.

It has been found possible to solve the problem in one case where the motion is not in one plane. If the electron moves in a circular helix with constant velocity so that $\xi = a \cos p\tau$, $\eta = a \sin p\tau$, $\zeta = d\tau$, where d is a constant, the Riccattian equation of § 1 may be written

$$2(c^2 - a^2 p^2 - d^2) \frac{d\sigma}{d\tau} = ap^2(d-c)e^{ip\tau} - 2i\sigma a^2 p^3 + ap^2 \sigma^2 e^{-ip\tau} (c+d).$$

Making use of the trial solution $Ae^{ip\tau}$ we find that σ_1, σ_2 are particular solutions where

$$(c+d)(1+\omega)\sigma_1 = iape^{ip\tau}, \quad (c+d)(1-\omega)\sigma_2 = iape^{ip\tau} \text{ and } (c^2-d^2)\omega^2 = c^2-d^2-a^2p^2.$$

Proceeding in the same way as before the general solution is found to be

$$\sigma = ie^{ip\tau} \left\{ \frac{(c-d)}{ap} (1+\omega) + \frac{1}{(\alpha+i\beta)e^{-ip\tau/\omega} - ap/2(c-d)\omega} \right\}$$

and from this on the work is the same as in the case of circular motion. In the case when $c > d$ and $c^2 < d^2 + a^2 p^2$ (so that ω is a pure imaginary) it is found as before that if we write $\omega = i\theta$ that $\lambda = 0$ if $\alpha\theta + \beta = 0$. Thus, again, there exists a single infinity of lines of force for which the directions of projections satisfy $\lambda = 0$. It has not been thought necessary to give the detailed results here as they are not essentially different from those already developed for the case of circular motion.

SECTION 5.

It was thought that it might be possible to find several curves with the same directions of projection. In other words, that, knowing a solution (l, m, n) of the equations (A) of § 1 for a definite curve of motion, it might be possible to find another curve or system of curves crossing the directions of projection for the original curve and such that these lines are also directions of projection for them. It was found that this is not in general the case and if a second path could be obtained in this way the three equations of the type

$$\left[p - \frac{2\lambda \{ (\bar{v} \bar{v}') + cp \}}{c^2 - v^2} \right] \frac{\partial l}{\partial \tau} = (\xi' - cl)q + \lambda \frac{\partial^2 l}{\partial \tau^2}$$

would be satisfied. From these it is easy to deduce that

$$\left(\frac{\partial l}{\partial \tau} \right)^2 + \left(\frac{\partial m}{\partial \tau} \right)^2 + \left(\frac{\partial n}{\partial \tau} \right)^2 = a\lambda^2$$

where a is a constant, an equation which is true in the case of rectilinear motion with uniform velocity.

CONCLUSION.

The main results of this paper may be restated as follows: The problem of finding the equations of the lines of electric force due to a moving electron, considered as a point charge, has been reduced to the calculation of an indefinite integral in the case when the motion is all in one plane and so may be regarded as solved. The more important cases of plane motion such as rectilinear motion and motion with constant velocity in a circle have been worked in detail and several lines have been found to have a more or less simple geometrical characterization; in particular, may be mentioned the line for the case of uniform circular motion which lies on a hyperboloid of revolution through the circle of motion and whose projection on the plane of this circle is an involute of the circle. The solution of the problem for the case of uniform motion in a circular helix has been indicated and this is interesting in so far that no general method has been found for cases where the motion of the point charge is not in one plane.

On Inequalities of Certain Types in General Linear Integral Equation Theory.

BY MARY EVELYN WELLS.

PART I.

§ 1. Introduction.

In the theory of the classical linear integral equation

$$\xi(s) = \eta(s) - \lambda \int_a^b x(s, t) \eta(t) dt$$

we find the inequality of Schwarz*

$$\int_a^b [\xi(s)]^2 ds \int_a^b [\eta(s)]^2 ds - \left[\int_a^b \xi(s) \eta(s) ds \right]^2 \geq 0.$$

In the theory of the general linear integral equation

$$\xi(s) = \eta(s) - \lambda Jx(s, t) \eta(t), \quad (1)$$

in which appear more general functions and operator, explained later, E. H. Moore has found, among other inequalities, the analogue of the Schwarz inequality given above, namely,

$$J\xi\bar{\xi}J\eta\bar{\eta} - J\xi\bar{\eta}J\eta\bar{\xi} \geq 0, \quad (2)$$

where — denotes the conjugate imaginary. We proceed to the discussion of such inequalities.

In a memoir "On the Foundations of the Theory of Linear Integral Equations",† E. H. Moore has given the basis Σ_s , and system of postulates, of a theory of the general linear integral equation (1) which we shall write

$$\xi = \eta - \lambda Jx\eta. \quad (1)$$

E. H. Moore has defined the properties of class $\mathfrak{M} = [\mu]$ of functions μ on a general range $\mathfrak{P} = [p]$ to the class $\mathfrak{A} = [a]$ of all real or complex numbers, properties of class $(\mathfrak{M}\mathfrak{M})_*$ to which the kernel function x belongs, and also the properties of the functional operation J , necessary for the theory of the gen-

* Heywood-Fréchet, "L'Equation de Fredholm."

† Bulletin of the American Mathematical Society, Ser. 2, Vol. XVIII.

eral linear integral equation. For the convenience of the reader some of the definitions are given here.

\mathfrak{M} is linear (L), in notation \mathfrak{M}^L , in case

$$\mathfrak{M} = \mathfrak{M}_L = [\text{all } \mu = a_1\mu_1 + a_2\mu_2 + \dots + a_n\mu_n].$$

\mathfrak{M} is real (R), in notation \mathfrak{M}^R , in case the class \mathfrak{M} is the same as the class of conjugate elements,

$$\mathfrak{M} = \overline{\mathfrak{M}}.$$

J is an operator of binary quality eliminating both the arguments s and t when it operates on such a function as $\mu_1(s)\mu_2(t)$ or $\kappa(s, t)$ giving a number of class \mathfrak{A} . That J operates on a function κ to give a number a , we indicate by the notation J on κ to \mathfrak{A} . The notation $J_{(s, t)}\kappa(s, t)$ will be replaced, throughout, by $J\kappa$, and $J_{(s, t)}\mu_1(s)\mu_2(t)$ by $J\mu_1\mu_2$. That $J\mu_1\mu_2$ is in general different from $J\mu_2\mu_1$ is seen by the examples of J used in § 4.

The operator J is linear (L), in notation J^L , in case

$$a_1\kappa_1 + a_2\kappa_2 = \kappa \text{ implies } a_1J\kappa_1 + a_2J\kappa_2 = J\kappa.$$

The operator J is hermitian (H), in notation J^H , in case

$$\overline{J\mu_1\mu_2} = J\bar{\mu}_2\bar{\mu}_1.$$

The operator J is positive (P), in notation J^P , in case for every function μ of \mathfrak{M}

$$J\bar{\mu}\mu \geq 0.$$

Using the foundation

$$\Sigma = (\mathfrak{A}; \mathfrak{B}; \mathfrak{M}^{LR}; \mathfrak{R} = (\mathfrak{M}\mathfrak{M})_*; J \text{ on } \mathfrak{R} \text{ to } \mathfrak{A} \cdot LHP),$$

it is the purpose of the first part of this paper to show the existence of certain inequalities of the type

$$\sum_{ijkl}^{(12)} a_{ijkl} J_{ij} J_{kl} \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; J^{LHP}); \quad (3)$$

where $ijkl$ is an arrangement of the digits 1234 which refer to the functions in the first, second, third and fourth positions in the product $\xi \bar{\xi} \eta \bar{\eta}$ of which the arguments are omitted for convenience. For instance the term $a_{1324} J_{13} J_{24} \xi \bar{\xi} \eta \bar{\eta}$ indicates $a_{1324} J \xi \eta J \bar{\xi} \bar{\eta}$. Of the twenty-four terms to correspond with the twenty-four arrangements of the subscripts 1234 in $J_{ij} J_{kl}$ only twelve are distinct since $J_{ij} J_{kl} = J_{kl} J_{ij}$. Thus, the inequality is one of twelve terms as indicated by the notation in (3). For definiteness the digit 1 will be kept in the first place or the last place in the arrangements of 1234, and the other digits will be in dictionary order. With this plan the coefficients of the twelve distinct

terms are as indicated in table (7). In further explanation of (3) it should be said that the parenthesis indicates that the inequalities are valid for all functions ξ and η of the class \mathfrak{M} and all operators which have the properties *LHP*. Since throughout this paper ξ and η will always be considered of the class \mathfrak{M} , and each J will have the properties *LHP*, these superscripts will often be omitted, and the superscripts used will usually indicate still more special properties of ξ , η , and J .

Further, it will be shown that the inequalities exhibited form a fundamental set for the two cases, (1°) where $J = \check{J}$, i. e., $J_{(s,t)}x(s,t) = J_{(s,t)}x(t,s)^*$, (2°) where ξ and η are real. By saying that certain inequalities form a fundamental set, it is meant that any inequality of type (3) which satisfies conditions shown to be necessary, can be expressed as a sum of positive or zero multiples of the inequalities forming the fundamental set.

The three following inequalities, containing a numerical parameter u , together with those obtainable from these three by transformations on ξ and η , form the fundamental set (1°) when $J = \check{J}$, (2°) when ξ and η are real.

$$\{(1 + u\bar{u})J_{12}J_{34} + (u + \bar{u})J_{14}J_{32}\}\xi\bar{\xi}\eta\bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; u^{\mathfrak{R}}; J^{LHP}). \quad (4)$$

$$\{J_{12}J_{43} + \bar{u}J_{14}J_{23} + uJ_{32}J_{41} + u\bar{u}J_{34}J_{21}\}\xi\bar{\xi}\eta\bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; u^{\mathfrak{R}}; J^{LHP}). \quad (5)$$

$$\{J_{13}J_{42} + \bar{u}J_{13}J_{24} + uJ_{42}J_{31} + u\bar{u}J_{24}J_{31}\}\xi\bar{\xi}\eta\bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; u^{\mathfrak{R}}; J^{LHP}). \quad (6)$$

§ 2. Proofs of the Inequalities.

A special case of the first inequality (4) is the generalized Schwarz inequality (2) stated and proved by E. H. Moore. The proof of inequality (4) was exhibited by E. H. Moore in class lectures given in January, 1914. For the derivation of (4) and (5) E. H. Moore proved that when a set of classes of functions $\mathfrak{M}', \mathfrak{M}'', \dots, \mathfrak{M}^{(n)}$, each of which is linear (*L*) and real (*R*), is used to form the class $\mathfrak{M} = (\mathfrak{M}'\mathfrak{M}'' \dots \mathfrak{M}^{(n)})_L$,† the resulting class is linear (*L*) and real (*R*); and that the corresponding functional operation $J = J'J'' \dots J^{(n)}$, in which each $J^{(i)}$ has the properties *LP*, itself has the properties *LP*. Accordingly, when $m=2$, we have

$$J'_{13}J''_{24}(\xi\eta + u\eta\xi)(\bar{\xi}\eta + u\eta\bar{\xi}) \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; u^{\mathfrak{R}}; J'^{LHP}; J''^{LHP}).$$

* E. H. Moore, *Bulletin of the American Mathematical Society*, April, 1912.

† "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations" published in the "Proceedings of the Fifth International Congress of Mathematicians, Cambridge, August, 1912."

Since the two operators are not necessarily the same, two inequalities are obtained, (4) by using JJ and (5) by using $J\bar{J}$.

To prove (6) it is only necessary to write the inequality as a product of a number and its conjugate:

$$(J\xi\eta + uJ\eta\xi)(\overline{J\xi\eta + uJ\eta\xi}) \geq 0 \quad (\xi^m; \eta^m; u^m; J^{LHP}).$$

Table (7), in which the inequalities are tabulated by means of their coefficients a_{ijk} and labeled A_1, A_2, \dots, A_{12} for convenience in reference, shows the complete set of inequalities (4), (5), (6) and those obtainable from (4), (5) and (6) by transformations on ξ and η . The transformations used may be indicated by the usual transformation notation: $^*(\xi\bar{\xi}), (\xi\bar{\xi})(\eta\bar{\eta}), (\eta\bar{\eta})$.

	a_{1234}	a_{1243}	a_{1324}	a_{1342}	a_{1423}	a_{1432}	a_{2341}	a_{2431}	a_{3241}	a_{3421}	a_{4231}	a_{4321}
$A_{1,u}$	$1+u\bar{u}$	0	0	0	0	$u+\bar{u}$	0	0	0	0	0	0
$A_{2,u}$	0	$1+u\bar{u}$	0	$u+\bar{u}$	0	0	0	0	0	0	0	0
$A_{3,u}$	0	0	0	0	0	0	$u+\bar{u}$	0	$1+u\bar{u}$	0	0	0
$A_{4,u}$	0	0	0	0	0	0	$u+\bar{u}$	0	0	0	$1+u\bar{u}$	0
$A_{5,u}$	0	1	0	0	\bar{u}	0	0	0	u	$u\bar{u}$	0	0
$A_{6,u}$	1	0	\bar{u}	0	0	0	0	0	0	0	u	$u\bar{u}$
$A_{7,u}$	0	$u\bar{u}$	0	0	\bar{u}	0	0	0	u	1	0	0
$A_{8,u}$	$u\bar{u}$	0	\bar{u}	0	0	0	0	0	0	0	u	1
$A_{9,u}$	0	0	\bar{u}	1	0	0	0	$u\bar{u}$	0	0	0	0
$A_{10,u}$	0	0	\bar{u}	$u\bar{u}$	0	0	0	1	0	0	u	0
$A_{11,u}$	0	0	0	0	\bar{u}	$u\bar{u}$	1	0	0	0	0	0
$A_{12,u}$	0	0	0	0	\bar{u}	1	$u\bar{u}$	0	u	0	0	0

§ 3. The Twelve Inequalities (7) Form a Fundamental Set When $J=\bar{J}$.

PROOF. Since the operator J is self-transpose ($J=\bar{J}$), $J\mu_1\mu_2=J\mu_2\mu_1$. As an example of such an operator may be mentioned the classical unary J of the classical instances II_n , III , IV ,† which, in the respective instances, is

$$\sum_{p=1}^n, \quad \sum_{p=1}^{\infty}, \quad \int_a^b dp.$$

When the self-transpose J operates upon $\xi\bar{\xi}\eta\bar{\eta}$, we have

$$\left. \begin{aligned} J_{12}J_{34} &= J_{12}J_{43} = J_{34}J_{21} = J_{43}J_{21}, \\ J_{13}J_{24} &= J_{13}J_{42} = J_{24}J_{31} = J_{42}J_{31}, \\ J_{14}J_{23} &= J_{14}J_{32} = J_{23}J_{41} = J_{32}J_{41}. \end{aligned} \right\} \quad (8)$$

* Cf. Cajori, "Theory of Equations."

† E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations."

Consequently, we have a grouping of coefficients a_{ijkl} which we designate

$$\left. \begin{aligned} b_1 &= a_{1234} + a_{1243} + a_{3421} + a_{4321}, \\ b_2 &= a_{1324} + a_{1342} + a_{2431} + a_{4231}, \\ b_3 &= a_{1423} + a_{1432} + a_{2341} + a_{3241}. \end{aligned} \right\} \quad (9)$$

In this notation the general inequality which we wish to build from the given inequalities (7) is

$$(b_1 J_{12} J_{34} + b_2 J_{13} J_{24} + b_3 J_{14} J_{23}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; J = \bar{J}). \quad (10)$$

In this field, $J = \bar{J}$, the given inequalities (7), with grouping of coefficients in accordance with (9), reduce to four distinct inequalities B_1, B_2, B_3, B_4 , tabulated by means of the coefficients b_1, b_2, b_3 , as follows:

	b_1	b_2	b_3
$B_{1,u}$	$1 + u\bar{u}$	0	$u + \bar{u}$
$B_{2,u}$	$1 + u\bar{u}$	$u + \bar{u}$	0
$B_{3,u}$	0	$1 + u\bar{u} + u + \bar{u}$	0
$B_{4,u}$	0	0	$1 + u\bar{u} + u + \bar{u}$

(11)

To build the general inequality (10), as the sum of positive or zero multiples of the fundamental inequalities (11), it is evident that the multipliers must be expressed in terms of the coefficients b_1, b_2 , and b_3 . First, then, we determine certain necessary conditions involving b_1, b_2 and b_3 , which are certain expressions in b_1, b_2, b_3 found to be necessarily positive or zero numbers. By suitable choice of certain of these expressions we build (10) as a sum of positive or zero multiples of the inequalities (11).

For the determination of necessary conditions on b_1, b_2, b_3 we need only to use a binary operator, and use for ξ and η the vectors (x_1, x_2) and (y_1, y_2) . In this binary algebraic case, the linearity of J demands that $J\xi\eta$ have the form

$$j_{11}x_1y_1 + j_{12}x_1y_2 + j_{21}x_2y_1 + j_{22}x_2y_2.$$

Therefore, the binary operator may be written

$$J = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

which effects the ordinary matricial combination with the product $\xi\eta$ which is

$$\begin{pmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{pmatrix}.$$

And, conversely, this form of J implies the linearity of J .

The fact that this operator is hermitian $(H)^*$ is expressed in the equality

$$\overline{j_{11}x_1y_1 + j_{12}x_1y_2 + j_{21}x_2y_1 + j_{22}x_2y_2} = j_{11}\overline{x_1y_1} + j_{21}\overline{x_1y_2} + j_{12}\overline{x_2y_1} + j_{22}\overline{x_2y_2} \quad (x_1, x_2, y_1, y_2),$$

for which it is necessary and sufficient that the matrix (j_{rs}) be hermitian, viz., $\bar{j}_{rs} = j_{sr}$, $(r=1, 2; s=1, 2)$.

The property P of J^* demands

$$j_{11}x_1\bar{x}_1 + j_{12}x_1\bar{x}_2 + j_{21}x_2\bar{x}_1 + j_{22}x_2\bar{x}_2 \geq 0 \quad (x_1, x_2).$$

For the positiveness of this hermitian form the necessary and sufficient conditions are

$$j_{11} \geq 0, \quad j_{22} \geq 0, \quad j_{11}j_{22} - j_{12}j_{21} \geq 0.$$

Hence the matrix (j_{rs}) is both hermitian and positive. In general, it is true that such an operator, binary or n -ary, is hermitian (H) if, and only if, the matrix (j_{rs}) is hermitian; and such an operator is positive (P) if, and only if, each principal minor is positive or zero.

The binary operator $\begin{pmatrix} j_{11} & 0 \\ 0 & j_{22} \end{pmatrix}$ affords, as a special instance of (10),

$$\left. \begin{aligned} & b_1(j_{11}x_1\bar{x}_1 + j_{22}x_2\bar{x}_2)(j_{11}y_1\bar{y}_1 + j_{22}y_2\bar{y}_2) \\ & + b_2(j_{11}x_1y_1 + j_{22}x_2y_2)(j_{11}\bar{x}_1\bar{y}_1 + j_{22}\bar{x}_2\bar{y}_2) \\ & + b_3(j_{11}x_1\bar{y}_1 + j_{22}x_2\bar{y}_2)(j_{11}\bar{x}_1y_1 + j_{22}\bar{x}_2y_2) \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} x_1, y_1, j_{11} \geq 0 \\ x_2, y_2, j_{22} \geq 0 \end{pmatrix}. \quad (12)$$

The cases

$$\begin{aligned} (\xi; \eta; J) &= (x_1, x_2; y_1, y_2; j_{11}, j_{22}) \\ &= (1, 0; 1, 0; 1, 0), (0, -1; 1, 0; 1, 1), (1, i; 1, -i; 1, 1), \\ &\quad (1, i; 1, i; 1, 1), \end{aligned}$$

where $i = \sqrt{-1}$, show that we have as necessary conditions:

$$b_1 + b_2 + b_3 \geq 0, \quad b_1 \geq 0, \quad b_1 + b_2 \geq 0, \quad b_1 + b_3 \geq 0. \quad (13)$$

With these conditions on the coefficients we are able at once to build the general inequality (10) in the field indicated by $J = \check{J}$ from the fundamental inequalities (11).

If $b_3 \geq 0$, we secure the desired inequality by using

$$\frac{1}{2}b_1B_{2,-1} + (b_1 + b_2)B_{3,0} + b_3B_{4,0},$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero. If $b_3 < 0$, we use

$$-\frac{1}{2}b_3B_{1,-1} + \frac{1}{2}(b_1 + b_3)B_{2,-1} + (b_1 + b_2 + b_3)B_{3,0},$$

which is the sum of positive or zero multiples of positive or zero forms, hence is positive or zero.

§ 4. *The Twelve Inequalities (7) Form a Fundamental Set When ξ and η Are Real.*

PROOF. In case ξ and η are real we have:

$$\left. \begin{aligned} J_{12}J_{34} &= J_{12}J_{43} = J_{34}J_{21} = J_{43}J_{21}, \\ J_{13}J_{42} &= J_{14}J_{32} = J_{23}J_{41} = J_{24}J_{31}, \\ J_{13}J_{24} &= J_{14}J_{23}, \\ J_{32}J_{41} &= J_{42}J_{31}. \end{aligned} \right\} \quad (14)$$

We shall designate the consequent grouping of coefficients a_{ijkl} ,

$$\left. \begin{aligned} c_1 &= a_{1234} + a_{1243} + a_{3421} + a_{4321}, \\ c_2 &= a_{1342} + a_{1432} + a_{2341} + a_{2431}, \\ c_3 &= a_{1324} + a_{1423}, \\ c_4 &= a_{3241} + a_{4231}. \end{aligned} \right\} \quad (15)$$

The corresponding general inequality which is to be expressed as the sum of positive or zero multiples of the fundamental inequalities is

$$(c_1 J_{12}J_{34} + c_2 J_{13}J_{42} + c_3 J_{13}J_{24} + c_4 J_{32}J_{41}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^R; \eta^R; J). \quad (16)$$

With ξ and η real, the given inequalities (7), with grouping of coefficients in accordance with (15), reduce to three distinct inequalities C_1, C_2, C_3 , tabulated by means of their coefficients c_1, c_2, c_3, c_4 as follows:

	c_1	c_2	c_3	c_4
$C_{1,u}$	$1 + u\bar{u}$	$u + \bar{u}$	0	0
$C_{2,u}$	$1 + u\bar{u}$	0	\bar{u}	u
$C_{3,u}$	0	$1 + u\bar{u}$	\bar{u}	u

(17)

The multipliers in terms of c_1, c_2, c_3, c_4 with which to build the general inequality (16), as a sum of positive or zero multiples of known inequalities (17) are to be chosen from necessary conditions involving the coefficients c_1, c_2, c_3, c_4 . Again, it is possible to determine the necessary conditions by use of a binary operator; but in order to secure sufficient conditions, a more general binary operator must be used than served for the situation $J = \check{J}$. We use j_{12} pure imaginary (I) and have

$$J = \begin{pmatrix} j_{11} & j_{12} \\ -j_{12} & j_{22} \end{pmatrix}.$$

The inequality (16) becomes, for this special case,

$$\left. \begin{aligned} & c_1(j_{11}x_1^2 + j_{22}x_2^2)(j_{11}y_1^2 + j_{22}y_2^2) \\ & + c_2\{j_{11}x_1y_1 + j_{12}(x_1y_2 - x_2y_1) + j_{22}x_2y_2\} \\ & \quad \{j_{11}x_1y_1 + j_{12}(x_2y_1 - x_1y_2) + j_{22}x_2y_2\} \\ & + c_3\{j_{11}x_1y_1 + j_{12}(x_1y_2 - x_2y_1) + j_{22}x_2y_2\}^2 \\ & + c_4\{j_{11}x_1y_1 + j_{12}(x_2y_1 - x_1y_2) + j_{22}x_2y_2\}^2 \end{aligned} \right\} \geq 0 \begin{pmatrix} x_1^R & y_1^R \\ x_2^R & y_2^R \\ j_{11} \geq 0 & \\ j_{22} \geq 0 & j_{12}^R \\ j_{11}j_{22} + j_{12}^2 \geq 0 \end{pmatrix}. \quad (18)$$

By making suitable choice of the variables and elements of the matrix J , we secure the necessary conditions. In this instance it is convenient to use the most general matrix whose determinant is zero. Such a matrix is

$$\begin{pmatrix} j_1 \bar{j}_1 & j_1 \bar{j}_2 \\ j_2 \bar{j}_1 & j_2 \bar{j}_2 \end{pmatrix} \text{ where } j_2 = e i j_1 \text{ and } e^R. \quad (19)$$

In this work we need frequently to remember that

$$me^2 + 2ne + p \geq 0 \quad (e^R) \quad (20)$$

implies

$$m^R, n^R, p^R, \quad m \geq 0, p \geq 0, \quad mp - n^2 \geq 0. \quad (21)$$

The cases $(x_1, x_2; y_1, y_2; j_{11}, j_{12}, j_{22})$

$$= (1, 0; 1, 0; 1, 0, 1), (1, 0; 0, 1; 1, i, 1), (1, 1; 1, -1; 1, 0, 1)$$

give as necessary conditions

$$\text{hence, } \left. \begin{aligned} & c_1 + c_2 + c_3 + c_4 \geq 0, \quad c_1 + c_2 - c_3 - c_4 \geq 0, \quad c_1 \geq 0, \\ & (c_1 + c_2)^R \geq 0, \quad (c_3 + c_4)^R, \quad c_1^R, c_2^R. \end{aligned} \right\} \quad (22)$$

The case $(x_1, x_2; y_1, y_2; j_{11}, j_{12}, j_{22}) = (1, 0; 1, 1; 1, -e'i, e'^2)$ gives the inequality

$$(c_1 + c_2 - c_3 - c_4)e'^2 - 2i(c_3 - c_4)e' + (c_1 + c_2 + c_3 + c_4) \geq 0 \quad (e'^R)$$

which, by (21), ensures the conditions

$$(c_3 - c_4)^R, \quad \therefore c_3 = \bar{c}_4, \quad \text{and} \quad (c_1 + c_2)^2 - 4c_3c_4 \geq 0. \quad (23)$$

With the conditions expressed in (22) and (23) it is possible to build the general inequality (16) in this field $(\xi^R; \eta^R)$, from the fundamental inequalities (17).

If $c_2 > 0$, we may build first for the case $c_1 = 0$, using the consequent condition:

$$c_2^2 - 4c_3c_4 \geq 0.$$

The desired inequality is given by

$$\frac{1}{2} \{ c_2 + (c_2^2 - 4c_3c_4)^{\frac{1}{2}} \} C_{3, 2c_4/[c_2 + (c_2^2 - 4c_3c_4)^{\frac{1}{2}}]}. \quad (24)$$

If $c_1 \neq 0$ it follows from (22) that $c_1 > 0$, and we may obtain the desired inequality by adding $c_1 C_{2,0}$ to the inequality (24) already built.

If $c_2 = 0$ the necessary conditions (22) and (23) reduce to

$$c_1 \geq 0, \quad c_1^2 - 4c_3c_4 \geq 0.$$

Hence, the desired inequality may be expressed as

$$\frac{1}{2} \{c_1 + (c_1^2 - 4c_3c_4)^{\frac{1}{2}}\} C_{2, 2c_4/[c_1 + (c_1^2 - 4c_3c_4)^{\frac{1}{2}}]}.$$

If $c_2 < 0$ and $c_1 + c_2 \neq 0$, we use

$$-\frac{1}{2}c_2 C_{1,-1} + d C_{2, c_4/d}, \quad \text{where } d = \frac{1}{2}[c_1 + c_2 + \{(c_1 + c_2)^2 - 4c_3c_4\}^{\frac{1}{2}}].$$

If $c_1 + c_2 = 0$, it follows that $c_3 = c_4 = 0$. Hence, we secure the desired inequality by using $\frac{1}{2} c_1 C_{1,-1}$.

PART II.

§ 5. Definition of Polarizable Inequality.

The inequality

$$\sum_{ijkl}^{(12)} a_{ijkl} J_{ij} J_{kl} \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; J^{LHP})$$

is said to be polarizable if it is true that

$$\sum_{ijkl}^{(12)} a_{ijkl} (J'_{ij} J''_{kl} + J''_{ij} J'_{kl}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^{\mathfrak{M}}; \eta^{\mathfrak{M}}; J'^{LHP}; J''^{LHP}). \quad (25)$$

It will be shown in this part of the paper that the first eight inequalities of (7) are polarizable, while the last four are not polarizable, even for the case $J = \bar{J}$, or for the case ξ and η real. Also, it will be shown that these eight polarizable inequalities (4') and (5') form a fundamental set of polarizable inequalities (1°) when $J = \bar{J}$, and (2°) when ξ and η are real.

§ 6. Proof of the Polarized Forms.

In § 2 we have

$$J'_{13} J''_{24} (\xi \eta + u \eta \xi) \overline{(\xi \eta + u \eta \xi)} \geq 0 \quad (\xi; \eta; u^{\mathfrak{M}}; J'; J'').$$

We may equally well have

$$J''_{13} J'_{24} (\xi \eta + u \eta \xi) \overline{(\xi \eta + u \eta \xi)} \geq 0 \quad (\xi; \eta; u^{\mathfrak{M}}; J'; J'').$$

The sum of these two inequalities is

$$\{ (1 + u\bar{u}) (J'_{12} J''_{34} + J''_{12} J'_{34}) + (u + \bar{u}) (J'_{14} J''_{32} + J''_{14} J'_{32}) \} \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; u^{\mathfrak{M}}; J'; J''). \quad (26)$$

This is the polarized form of (4). Similarly, by using $J' \bar{J}''$ and $J'' \bar{J}'$ we obtain the polarized form of (5). Transformations* on ξ and η afford the polarized

* Cf. Section 2.

forms of the remaining six inequalities of (4') and (5'). It will be seen from conditions (30) and (34) that the inequalities (6') are not polarizable even when J is self-transpose or ξ and η are real.

§ 7. *The Eight Polarizable Inequalities (4') and (5') Form a Fundamental Set of Polarizable Inequalities When $J = \check{J}$.*

PROOF. Since $J' = \check{J}'$ and $J'' = \check{J}''$, and the operand is $\xi \bar{\xi} \eta \bar{\eta}$, it follows that

$$\left. \begin{aligned} J'_{12} J''_{34} &= J'_{12} J''_{43} = J'_{21} J''_{34} = J'_{21} J''_{43}, \\ J'_{34} J''_{12} &= J'_{34} J''_{21} = J'_{43} J''_{12} = J'_{43} J''_{21}, \\ J'_{13} J''_{24} &= J'_{13} J''_{42} = J'_{31} J''_{24} = J'_{31} J''_{42}, \\ J'_{24} J''_{13} &= J'_{24} J''_{31} = J'_{42} J''_{13} = J'_{42} J''_{31}, \\ J'_{14} J''_{23} &= J'_{14} J''_{32} = J'_{41} J''_{23} = J'_{41} J''_{32}, \\ J'_{23} J''_{14} &= J'_{23} J''_{41} = J'_{32} J''_{14} = J'_{32} J''_{41}. \end{aligned} \right\} \quad (27)$$

The corresponding grouping of coefficients a_{ijkl} is

$$a_{1234} + a_{1243} + a_{3421} + a_{4321}, \quad a_{1324} + a_{1342} + a_{2431} + a_{4231}, \quad a_{1423} + a_{1432} + a_{2341} + a_{3241},$$

which have already been defined (9) as b_1, b_2, b_3 , respectively.

The general inequality which we wish to build from the eight given inequalities (4') and (5') is

$$(b_1 J_{12} J_{34} + b_2 J_{13} J_{24} + b_3 J_{14} J_{23}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; J = \check{J}). \quad (10)$$

However, since this inequality is polarizable, it must be true that

$$\{b_1 (J'_{12} J''_{34} + J'_{34} J''_{12}) + b_2 (J'_{13} J''_{24} + J'_{24} J''_{13}) + b_3 (J'_{14} J''_{23} + J'_{23} J''_{14})\} \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; J' = \check{J}'; J'' = \check{J}''). \quad (28)$$

Using the operators

$$J' = \begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22} \end{pmatrix} \quad \text{and} \quad J'' = \begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22} \end{pmatrix},$$

we have as a special instance of (28),

$$\left. \begin{aligned} &b_1 \left\{ \begin{aligned} &(j'_{11} x_1 \bar{x}_1 + j'_{22} x_2 \bar{x}_2) (j''_{11} y_1 \bar{y}_1 + j''_{22} y_2 \bar{y}_2) \\ &+ (j'_{11} y_1 \bar{y}_1 + j'_{22} y_2 \bar{y}_2) (j''_{11} x_1 \bar{x}_1 + j''_{22} x_2 \bar{x}_2) \end{aligned} \right\} \\ &+ b_2 \left\{ \begin{aligned} &(j'_{11} x_1 y_1 + j'_{22} x_2 y_2) (j''_{11} \bar{x}_1 \bar{y}_1 + j''_{22} \bar{x}_2 \bar{y}_2) \\ &+ (j'_{11} \bar{x}_1 \bar{y}_1 + j'_{22} \bar{x}_2 \bar{y}_2) (j''_{11} x_1 y_1 + j''_{22} x_2 y_2) \end{aligned} \right\} \\ &+ b_3 \left\{ \begin{aligned} &(j'_{11} x_1 y_1 + j'_{22} x_2 y_2) (j''_{11} \bar{x}_1 y_1 + j''_{22} \bar{x}_2 y_2) \\ &+ (j'_{11} \bar{x}_1 y_1 + j'_{22} \bar{x}_2 y_2) (j''_{11} x_1 \bar{y}_1 + j''_{22} x_2 \bar{y}_2) \end{aligned} \right\} \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11} \geq 0 & j'_{22} \geq 0 \\ j''_{11} \geq 0 & j''_{22} \geq 0 \end{pmatrix}. \quad (29)$$

The cases $(x_1, x_2; y_1, y_2; j'_{11}, j'_{22}, j''_{11}, j''_{22})$

$$\begin{aligned} &= (1, 0; 1, 0; 1, 0; 1, 0), \quad (1, i; 1, -i; 0, 1; 1, 0), \\ &\quad (1, i; 1, i; 0, 1; 1, 0), \quad (-1, 1; 1, 1; 0, 1; 1, 0), \end{aligned}$$

show that we have as necessary conditions

$$b_1 + b_2 + b_3 \geq 0, \quad b_1 + b_2 - b_3 \geq 0, \quad b_1 - b_2 + b_3 \geq 0, \quad b_1 - b_2 - b_3 \geq 0. \quad (30)$$

Since all polarizable inequalities for the field $J = \tilde{J}$ must satisfy condition (30), and since neither $B_{3,u}$ nor $B_{4,u}$ of (11) satisfies the last condition of (30), it is now seen, as was suggested in § 5, that neither $B_{3,u}$ nor $B_{4,u}$ is polarizable. It follows that the inequalities (6') from which $B_{3,u}$ and $B_{4,u}$ were obtained do not satisfy the definition of polarizable inequality, as that definition demands that the polarized form be positive or zero for all J^{LHP} , of which we have $J = \tilde{J}^{LHP}$ as an instance.

The eight inequalities (4') and (5') which were proved polarizable, reduce to two when $J = \tilde{J}$, designated as $B_{1,u}$ and $B_{2,u}$ under (11).

The desired inequality (10) is shown to be the sum of positive or zero multiples of the fundamental polarizable inequalities $B_{1,u}$ and $B_{2,u}$ as follows:

When $b_3 \geq 0$ and $b_1 \neq b_3$, the desired inequality has the form

$$\frac{1}{2} b_3 B_{1,1} + \frac{1}{2} d_1 B_{2,b_2/d_1},$$

where

$$d_1 = b_1 - b_3 + \frac{1}{2} (b_1 - b_3)^2 - b_2^2 \frac{1}{2}.$$

When $b_3 < 0$ and $b_1 + b_3 \neq 0$, the desired inequality has the form

$$-\frac{1}{2} b_3 B_{1,-1} + \frac{1}{2} d_2 B_{2,b_2/d_2},$$

where

$$d_2 = b_1 + b_3 + \frac{1}{2} (b_1 + b_3)^2 - b_2^2 \frac{1}{2}.$$

When $b_1 = b_3$, we use $\frac{1}{2} b_1 B_{1,1}$, since $b_2 = 0$. When $b_1 = -b_3$, we use $\frac{1}{2} b_1 B_{1,-1}$, since $b_2 = 0$.

§ 8. The Eight Polarizable Inequalities (4') and (5') Form a Fundamental Set When ξ and η Are Real.

PROOF. In this instance (ξ^R, η^R)

$$\left. \begin{aligned} J'_{12} J''_{34} &= J'_{12} J''_{43} = J'_{21} J''_{34} = J'_{21} J''_{43}, \\ J'_{34} J''_{21} &= J'_{43} J''_{21} = J'_{34} J''_{12} = J'_{43} J''_{12}, \\ J'_{13} J''_{24} &= J'_{14} J''_{23} = J'_{24} J''_{13} = J'_{23} J''_{14}, \\ J'_{13} J''_{42} &= J'_{14} J''_{32} = J'_{23} J''_{41} = J'_{24} J''_{31}, \\ J'_{42} J''_{13} &= J'_{32} J''_{14} = J'_{41} J''_{23} = J'_{31} J''_{24}, \\ J'_{32} J''_{41} &= J'_{42} J''_{31} = J'_{41} J''_{32} = J'_{31} J''_{42}. \end{aligned} \right\} \quad (31)$$

The corresponding grouping of coefficients is that indicated in (15). Hence the inequality (16) is the general inequality which is to be expressed as the

sum of positive or zero multiples of the known polarizable inequalities, $C_{1,u}$ and $C_{2,u}$, to which (4') and (5') reduce when ξ and η are real. The positive or zero multipliers will be chosen from necessary conditions on c_1, c_2, c_3, c_4 . The conditions (22) and (23), which must be satisfied by the coefficients of all inequalities (16) must necessarily be satisfied by the coefficients of the polarizable inequality (16). In addition to these conditions (22) and (23), necessary conditions are obtained by using the fact that the inequality (16) is polarizable; i. e.

$$\{c_1(J'_{12}J''_{34} + J'_{34}J''_{12}) + c_2(J'_{13}J''_{42} + J'_{42}J''_{13}) + c_3(J'_{13}J''_{24} + J'_{24}J''_{13}) + c_4(J'_{32}J''_{41} + J'_{41}J''_{32})\} \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^R; \eta^R; J'; J''). \quad (32)$$

The use of

$$J' = \begin{pmatrix} j'_{11} & j'_{12} \\ -j'_{12} & j'_{22} \end{pmatrix}, \quad J'' = \begin{pmatrix} j''_{11} & j''_{12} \\ -j''_{12} & j''_{22} \end{pmatrix},$$

where j'_{12} and j''_{12} are pure imaginary, gives as a special case of (32):

$$\left. \begin{aligned} & c_1 \{ (j'_{11}x_1^2 + j'_{22}x_2^2) (j''_{11}y_1^2 + j''_{22}y_2^2) \\ & \quad + (j'_{11}y_1^2 + j'_{22}y_2^2) (j''_{11}x_1^2 + j''_{22}x_2^2) \} \\ & + c_2 \left[\begin{aligned} & \{ j'_{11}x_1y_1 + j'_{12}(x_1y_2 - x_2y_1) + j'_{22}x_2y_2 \} \\ & \{ j''_{11}x_1y_1 + j''_{12}(x_2y_1 - x_1y_2) + j''_{22}x_2y_2 \} \\ & + \{ j'_{11}x_1y_1 + j'_{12}(x_2y_1 - x_1y_2) + j'_{22}x_2y_2 \} \\ & \{ j''_{11}x_1y_1 + j''_{12}(x_1y_2 - x_2y_1) + j''_{22}x_2y_2 \} \end{aligned} \right] \\ & + 2c_3 \{ j'_{11}x_1y_1 + j'_{12}(x_1y_2 - x_2y_1) + j'_{22}x_2y_2 \} \\ & \quad \{ j''_{11}x_1y_1 + j''_{12}(x_1y_2 - x_2y_1) + j''_{22}x_2y_2 \} \\ & + 2c_4 \{ j'_{11}x_1y_1 + j'_{12}(x_2y_1 - x_1y_2) + j'_{22}x_2y_2 \} \\ & \quad \{ j''_{11}x_1y_1 + j''_{12}(x_2y_1 - x_1y_2) + j''_{22}x_2y_2 \} \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} x_1^R & x_2^R \\ y_1^R & y_2^R \\ j'_{11} \geq 0 & j'_{22} \geq 0 \\ j''_{11} \geq 0 & j''_{22} \geq 0 \\ j'_{12} & j''_{12} \\ j'_{11}j'_{22} + j'^2_{12} \geq 0 \\ j''_{11}j''_{22} + j''^2_{12} \geq 0 \end{pmatrix}. \quad (33)$$

The special values

$$(x_1, x_2; y_1, y_2; j'_{11}, j'_{12}, j'_{22}; j''_{11}, j''_{12}, j''_{22}) \\ = (1, 0; 0, 1; 1, i, 1; 1, -i, 1), (1, 1; 1, -1; 0, 0, 1; 1, 0, 0)$$

give the necessary conditions

$$\left. \begin{aligned} & c_1 - c_2 + c_3 + c_4 \geq 0, \quad c_1 - c_2 - c_3 - c_4 \geq 0, \\ & c_1 - c_2 \geq 0. \end{aligned} \right\} \quad \text{whence} \quad (34)$$

Again, it is convenient to use, for J , a matrix whose determinant is zero. We choose

$$J' = \begin{pmatrix} j'_1 \bar{j}'_1 & j'_1 \bar{j}'_2 \\ j'_2 \bar{j}'_1 & j'_2 \bar{j}'_2 \end{pmatrix}, \quad J'' = \begin{pmatrix} j''_1 \bar{j}''_1 & j''_1 \bar{j}''_2 \\ j''_2 \bar{j}''_1 & j''_2 \bar{j}''_2 \end{pmatrix},$$

where $j'_2 = e'ij'_1$ and $j''_2 = e''ij''_1$ with e' and e'' real. Then (33) becomes a quadratic in e' and also in e'' . As an instance we set

$(x_1, x_2; y_1, y_2; j'_{11}, j'_{12}, j'_{22}; j''_{11}, j''_{12}, j''_{22}) = (1, 1; 1, -1; 1, -e'i, e'^2; 1, -e''i, e''^2)$, and (33) gives the quadratic expression $P(e', e'')$ in e' and e'' which must be positive or zero for every real e' and e'' , and for which, therefore, the discriminant must be positive or zero.

$$P(e', e'') = 2c_1(1+e'^2)(1+e''^2) + c_2 \left\{ \begin{aligned} &(1+2e'i-e'^2)(1-2e''i-e''^2) \\ &+ (1-2e'i-e'^2)(1+2e''i-e''^2) \end{aligned} \right\} + 2c_3(1+2e'i-e'^2)(1+2e''i-e''^2) + 2c_4(1-2e'i-e'^2)(1-2e''i-e''^2) \geq 0 \quad (e'^R, e''^R).$$

The discriminant of $P(e', 0)$ gives the condition

$$c_1^2 - (c_2 + c_3 + c_4)^2 + (c_3 - c_4)^2 \geq 0;$$

the discriminant of $P(e', 1)$ gives

$$c_1^2 - (c_2 - c_3 - c_4)^2 + (c_3 - c_4)^2 \geq 0,$$

whence

$$c_1^2 - c_2^2 - 4c_3c_4 \geq 0,$$

and, thus,

$$c_1^2 + c_2^2 - 4c_3c_4 \geq 0. \quad (35)$$

The discriminant of $P(e' e'')$ as to e'' is a homogeneous quadratic expression in $(1-e'^2, 2e')$, and since for e' real, even between the values -1 and $+1$, $2e'/(1-e'^2)$ takes every real value, the discriminant of this homogeneous expression must be positive or zero. Hence, we have the condition

$$(c_1^2 + c_2^2 - 4c_3c_4)^2 - 4c_1^2c_2^2 \geq 0,$$

and, therefore, by virtue of (35)

$$c_1^2 + c_2^2 - 4c_3c_4 \geq \pm 2c_1c_2$$

or

$$(c_1 \pm c_2)^2 - 4c_3c_4 \geq 0. \quad (36)$$

It can be seen from (34) that $C_{3,u}$ of (17) is not polarizable. By use of (22), (23), (34) and (36) we build all polarizable inequalities in this field $(\xi^R; \eta^R)$ as the sum of positive or zero multiples of the polarizable inequalities $C_{1,u}$ and $C_{2,u}$ of (17) in the following ways:

If $c_2 \geq 0$ and $c_1 \neq c_2$, the desired inequality may be expressed as

$$\frac{1}{2} c_2 C_{1,1} + d_1 C_{2, c_1/d_1},$$

where

$$d_1 = \frac{1}{2} [c_1 - c_2 + \{(c_1 - c_2)^2 - 4c_3c_4\}^{\frac{1}{2}}].$$

If $c_2 < 0$ and $c_1 + c_2 \neq 0$, the desired inequality has the form

$$-\frac{1}{2}c_2C_{1,-1} + d_2C_{2,c_4/d_2},$$

where

$$d_2 = \frac{1}{2}[c_1 + c_2 + \{(c_1 + c_2)^2 - 4c_3c_4\}^{1/2}].$$

If $c_2 \geq 0$ and $c_1 = c_2$, it follows from (36) that $c_3 = c_4 = 0$, and we may express the inequality as

$$\frac{1}{2}c_1C_{1,1}.$$

If $c_2 < 0$ and $c_1 + c_2 = 0$, it follows from (36) that $c_3 = c_4 = 0$, and we may use as the desired inequality

$$\frac{1}{2}c_1C_{1,-1}.$$

PART III.

§ 9. Bilinear Inequalities.

Somewhat related to the preceding problem is that of determining all bilinear inequalities of the form

$$\sum_{ijkl}^{(24)} z_{ijkl} J'_i J''_j \bar{\xi}_k \bar{\eta}_l \geq 0 \quad (\xi; \eta; J'; J'').$$

It is the purpose of this portion of the paper to exhibit sixteen such inequalities and prove that they form a fundamental set of bilinear inequalities for the cases (1°) $J = \bar{J}$ and (2°) ξ and η real functions.

We have, in § 2, as a special instance of the theorem proved by E. H. Moore, the inequality

$$J'_{13} J''_{24} (\xi\eta + u\eta\xi) (\bar{\xi}\bar{\eta} + u\bar{\eta}\bar{\xi}) \geq 0 \quad (\xi; \eta; u^{\Re}; J'; J''),$$

which is also written

$$(J'_{12} J''_{34} + \bar{u} J'_{14} J''_{32} + u J'_{32} J''_{14} + u\bar{u} J'_{34} J''_{12}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; u^{\Re}; J'; J'').$$

By using J' instead of J , J'' instead of J'' , and J' and J'' instead of J and J'' we have also

$$(J'_{21} J''_{34} + \bar{u} J'_{41} J''_{32} + u J'_{23} J''_{14} + u\bar{u} J'_{43} J''_{12}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; u^{\Re}; J'; J''),$$

$$(J'_{12} J''_{43} + \bar{u} J'_{14} J''_{23} + u J'_{32} J''_{41} + u\bar{u} J'_{34} J''_{21}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; u^{\Re}; J'; J''),$$

$$(J'_{21} J''_{43} + \bar{u} J'_{41} J''_{23} + u J'_{23} J''_{41} + u\bar{u} J'_{43} J''_{21}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; u^{\Re}; J'; J'').$$

Four more inequalities may be written by interchange of J' and J'' , the eight being $Z_{1,u}$ to $Z_{8,u}$ inclusive, tabulated according to their coefficients z_{ijkl} in (37). By replacing ξ by $\bar{\xi}$ the inequalities $Z_{9,u}, \dots, Z_{16,u}$ of (37) are obtained from inequalities $Z_{1,u}, \dots, Z_{8,u}$.

	z_{1234}	z_{1243}	z_{1324}	z_{1342}	z_{1423}	z_{1432}	z_{2341}	z_{2314}	z_{2431}	z_{2413}	z_{3421}	z_{3412}	z_{4321}	z_{4312}	z_{4231}	z_{4213}	z_{4123}	z_{4132}	z_{3124}	z_{3142}	z_{2134}	z_{2143}		
$Z_{1,u}$	1	0	0	0	0	\bar{u}	0	0	0	0	0	0	$u\bar{u}$	0	0	0	0	u	0	0	0	0		
$Z_{2,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	0	u	0	0	0	\bar{u}	1	0	0
$Z_{3,u}$	0	1	0	0	\bar{u}	0	0	0	u	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	
$Z_{4,u}$	0	0	0	0	0	0	u	0	0	0	0	0	$u\bar{u}$	0	0	0	0	0	\bar{u}	0	0	0	1	
$Z_{5,u}$	$u\bar{u}$	0	0	0	0	u	0	0	0	0	0	0	1	0	0	0	0	\bar{u}	0	0	0	0	0	
$Z_{6,u}$	0	$u\bar{u}$	0	0	u	0	0	0	\bar{u}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	
$Z_{7,u}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	\bar{u}	0	0	0	u	$u\bar{u}$	0	0
$Z_{8,u}$	0	0	0	0	0	0	\bar{u}	0	0	0	0	1	0	0	0	0	0	0	u	0	0	0	$u\bar{u}$	
$Z_{9,u}$	0	0	0	0	0	0	0	\bar{u}	0	$u\bar{u}$	0	0	0	0	0	0	0	0	u	0	1	0	0	
$Z_{10,u}$	0	1	0	\bar{u}	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	
$Z_{11,u}$	0	0	0	0	0	0	0	u	0	1	0	0	0	0	0	0	0	0	\bar{u}	0	$u\bar{u}$	0	0	
$Z_{12,u}$	0	$u\bar{u}$	0	u	0	0	0	0	0	0	0	0	0	1	0	\bar{u}	0	0	0	0	0	0	0	
$Z_{13,u}$	1	0	u	0	0	0	0	0	0	0	\bar{u}	$u\bar{u}$	0	0	0	0	0	0	0	0	0	0	0	
$Z_{14,u}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$u\bar{u}$	0	u	0	0	0	0	0	\bar{u}	1	
$Z_{15,u}$	$u\bar{u}$	0	\bar{u}	0	0	0	0	0	0	0	0	0	u	1	0	0	0	0	0	0	0	0	0	
$Z_{16,u}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	\bar{u}	0	0	0	0	0	u	$u\bar{u}$		

(37)

(37)

§ 10. The Sixteen Inequalities (37) Form a Fundamental Set of Bilinear Inequalities When $J' = \bar{J}'$ and $J'' = \bar{J}''$.

PROOF. In this field, $J' = \bar{J}'$ and $J'' = \bar{J}''$, equations (27) are valid, and they indicate the new coefficients,

$$\left. \begin{aligned} w_1 &= z_{1234} + z_{1243} + z_{2134} + z_{2143}, & w_2 &= z_{1324} + z_{1342} + z_{3124} + z_{3142}, \\ w_3 &= z_{1423} + z_{1432} + z_{4123} + z_{4132}, & w_4 &= z_{2341} + z_{2314} + z_{3214} + z_{3241}, \\ w_5 &= z_{2431} + z_{2413} + z_{4231} + z_{4213}, & w_6 &= z_{3421} + z_{3412} + z_{4312} + z_{4321} \end{aligned} \right\} \quad (38)$$

Accordingly the general inequality which we desire to build is

$$(w_1 J'_{12} J''_{34} + w_2 J'_{13} J''_{24} + w_3 J'_{14} J''_{23} + w_4 J'_{23} J''_{41} + w_5 J'_{24} J''_{31} + w_6 J'_{34} J''_{21}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi; \eta; J' = \bar{J}'; J'' = \bar{J}''). \quad (39)$$

The known inequalities from which (39) is to be built in this field, $J' = \bar{J}'$ and $J'' = \bar{J}''$, are the four inequalities to which (37) reduce on account of (38), and are tabulated in (40) according to their coefficients w_1, \dots, w_6 .

	w_1	w_2	w_3	w_4	w_5	w_6
$W_{1,u}$	1	0	\bar{u}	u	0	$u\bar{u}$
$W_{2,u}$	$u\bar{u}$	0	u	\bar{u}	0	1
$W_{3,u}$	1	u	0	0	\bar{u}	$u\bar{u}$
$W_{4,u}$	$u\bar{u}$	\bar{u}	0	0	u	1

(40)

As before, it is possible to secure the necessary conditions from the form corresponding to binary operators. Use of

$$J' = \begin{pmatrix} j'_{11} & 0 \\ 0 & j'_{22} \end{pmatrix}, \quad J'' = \begin{pmatrix} j''_{11} & 0 \\ 0 & j''_{22} \end{pmatrix},$$

gives as an instance of (39):

$$\left. \begin{aligned} &w_1(j'_{11}x_1\bar{x}_1 + j'_{22}x_2\bar{x}_2)(j''_{11}y_1\bar{y}_1 + j''_{22}y_2\bar{y}_2) \\ &+ w_2(j'_{11}x_1y_1 + j'_{22}x_2y_2)(j''_{11}\bar{x}_1\bar{y}_1 + j''_{22}\bar{x}_2\bar{y}_2) \\ &+ w_3(j'_{11}x_1\bar{y}_1 + j'_{22}x_2\bar{y}_2)(j''_{11}\bar{x}_1y_1 + j''_{22}\bar{x}_2y_2) \\ &+ w_4(j'_{11}\bar{x}_1y_1 + j'_{22}\bar{x}_2y_2)(j''_{11}x_1\bar{y}_1 + j''_{22}x_2\bar{y}_2) \\ &+ w_5(j'_{11}\bar{x}_1\bar{y}_1 + j'_{22}\bar{x}_2\bar{y}_2)(j''_{11}x_1y_1 + j''_{22}x_2y_2) \\ &+ w_6(j'_{11}y_1\bar{y}_1 + j'_{22}y_2\bar{y}_2)(j''_{11}x_1\bar{x}_1 + j''_{22}x_2\bar{x}_2) \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ j'_{11} \geq 0 & j'_{22} \geq 0 \\ j''_{11} \geq 0 & j''_{22} \geq 0 \end{pmatrix}. \quad (41)$$

The cases $(x_1, x_2; y_1, y_2; j'_{11}, j'_{22}; j''_{11}, j''_{22}) = (0, 1; 1, 0; 0, 1; 1, 0), (1, 0; 0, 1; 0, 1; 1, 0)$

give as necessary conditions,

$$w_1 \geq 0, \quad w_6 \geq 0. \quad (42)$$

The case $(x_1, x_2; y_1, y_2; j'_{11}, j'_{22}; j''_{11}, j''_{22}) = (1, k_1 + ik_2; 1, k_1 + ik_2; 0, 1; 1, 0)$, where k_1 and k_2 are real, gives as a special case of (41),

$$(w_1 + w_3 + w_4 + w_6)(k_1^2 + k_2^2) + (w_2 + w_5)(k_1^2 - k_2^2) + 2i(w_2 - w_5)k_1k_2 \geq 0 \quad (k_1^R, k_2^R). \quad (43)$$

In (43) the values $(k_1, k_2) = (1, 0), (0, 1)$ give the necessary conditions

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 \geq 0, \quad w_1 - w_2 + w_3 + w_4 - w_5 + w_6 \geq 0. \quad (44)$$

Hence, from (42) and (44),

$$(w_2 + w_5)^R, \quad (w_3 + w_4)^R,$$

while from (43) it is now seen that $w_2 - w_5$ is pure imaginary, and, therefore,

$$w_2 = \bar{w}_5. \quad (45)$$

Similarly, the case

$$(x_1, x_2; y_1, y_2; j'_{11}, j'_{22}; j''_{11}, j''_{22}) = (1, k_1 - ik_2; 1, k_1 + ik_2; 0, 1; 1, 0)$$

yields the conditions

$$w_1 + w_2 - w_3 - w_4 + w_5 + w_6 \geq 0, \quad (w_3 - w_4)^I,$$

whence

$$w_3 = \bar{w}_4. \quad (46)$$

The case $(x_1, x_2; y_1, y_2; j'_{11}, j'_{22}; j''_{11}, j''_{22}) = (1, k_1 + ik_2; 1, l_1 + il_2; 0, 1; 1, 0)$, where k_1, k_2, l_1, l_2 are real, gives as an instance of (41),

$$\left. \begin{aligned} &w_6 l_1^2 + l_1 \{ (w_2 + w_3 + w_4 + w_5) k_1 + i(w_2 + w_3 - w_4 - w_5) k_2 \} \\ &+ w_6 l_2^2 + l_2 \{ i(w_2 - w_3 + w_4 - w_5) k_1 - (w_2 - w_3 - w_4 + w_5) k_2 \} \\ &+ w_1 (k_1^2 + k_2^2) \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} k_1^R & k_2^R \\ l_1^R & l_2^R \end{pmatrix}. \quad (47)$$

Inequality (47) shows that $w_6=0$ demands:

$$\left. \begin{aligned} w_2 + w_3 + w_4 + w_5 &= 0, & w_2 + w_3 - w_4 - w_5 &= 0, \\ w_2 - w_3 + w_4 - w_5 &= 0, & w_2 - w_3 - w_4 + w_5 &= 0. \end{aligned} \right\} \quad (48)$$

For if not, suppose any one of (48) is not zero, say

$$w_2 + w_3 + w_4 + w_5 \neq 0.$$

Then by taking $(k_1, k_2, l_2) = (1, 0, 0)$ we have,

$$l_1(w_2 + w_3 + w_4 + w_5) + w_1 \geq 0, \quad (l_1^R),$$

which obviously is not true. Similarly, the other equations of (48) are proved when $w_6=0$. Also, equations (48) follow when $w_1=0$. Hence,

$$w_1=0 \text{ or } w_6=0 \text{ demands } w_2=w_3=w_4=w_5=0. \quad (49)$$

Thus for the remaining cases we may suppose $w_1 \neq 0$ and $w_6 \neq 0$ and omit w_1 or w_6 as a factor in any expression which is positive or zero, each of whose terms contains as factor one or more of the coefficients w_2, \dots, w_5 , or $w_1 w_6$.

When $l_1=0$, (47) becomes a quadratic in l_2 , whose discriminant is a quadratic in k_1 and k_2 , viz.:

$$\left\{ 4w_1 w_6 + (w_2 - w_3 + w_4 - w_5)^2 \{ k_1^2 + \{ 4w_1 w_6 - (w_2 - w_3 - w_4 + w_5)^2 \} k_2^2 \} + 2i(w_2 - w_3 + w_4 - w_5)(w_2 - w_3 - w_4 + w_5) k_1 k_2 \right\} \geq 0 \quad (k_1^R k_2^R).$$

The discriminant of this quadratic in k_1 and k_2 gives the condition

$$16w_1 w_6 \{ w_1 w_6 + (w_2 - w_3)(w_4 - w_5) \} \geq 0,$$

which by (49) gives as a condition always holding

$$w_1 w_6 + (w_2 - w_3)(w_4 - w_5) \geq 0. \quad (50)$$

Similarly, $l_2=0$ reduces (47) to a quadratic in l_1 whose discriminant is a quadratic in k_1 and k_2 , which gives the necessary condition

$$w_1 w_6 - (w_2 + w_3)(w_4 + w_5) \geq 0. \quad (51)$$

Conditions (50) and (51) combine to give

$$\left. \begin{aligned} w_1 w_6 - w_3 w_4 - w_2 w_5 &\geq 0, \\ w_1 w_6 + w_3 w_4 - w_2 w_5 &\geq 0, \\ w_1 w_6 - w_3 w_4 + w_2 w_5 &\geq 0. \end{aligned} \right\} \quad (52)$$

whence
and

The discriminant for (47) as a quadratic in l_1 is a quadratic in l_2 , viz.:

$$\left. \begin{aligned} &4w_6^2 l_2^2 + 4w_6 l_2 \{i(w_2 - w_3 + w_4 - w_5)k_1 - (w_2 - w_3 - w_4 + w_5)k_2\} \\ &\quad + \{4w_1 w_6 - (w_2 + w_3 + w_4 + w_5)^2\} k_1^2 \\ &\quad + \{4w_1 w_6 + (w_2 + w_3 - w_4 - w_5)^2\} k_2^2 \\ &\quad - 2i(w_2 + w_3 + w_4 + w_5)(w_2 + w_3 - w_4 - w_5)k_1 k_2 \end{aligned} \right\} \geq 0 \quad \begin{pmatrix} k_1^R & k_2^R \\ l_2^R \end{pmatrix},$$

whose discriminant, aside from a factor $16w_6^2$, reduces to

$$\left\{ \begin{aligned} &\{w_1 w_6 - (w_2 + w_4)(w_3 + w_5)\} k_1^2 + \{w_1 w_6 + (w_2 - w_4)(w_3 - w_5)\} k_2^2 \\ &\quad - 2i(w_2 w_3 - w_4 w_5)k_1 k_2 \end{aligned} \right\} \geq 0 \quad (k_1^R, k_2^R).$$

Hence,

$$\begin{aligned} &\{w_1 w_6 - (w_2 + w_4)(w_3 + w_5)\} \{w_1 w_6 + (w_2 - w_4)(w_3 - w_5)\} + (w_2 w_3 - w_4 w_5)^2 \geq 0, \\ \text{or} \quad &(w_1 w_6 - w_3 w_4 - w_2 w_5)^2 - 4w_2 w_3 w_4 w_5 \geq 0. \end{aligned} \quad (53)$$

Apart from exceptional cases, we have the desired general inequality (39) in the form

$$\frac{d_1}{2w_6} W_{1, 2w_6 w_4/d_1} + \frac{d_2}{2w_1} W_{4, 2w_1 w_5/d_2},$$

where

$$\begin{aligned} d_1 &= w_1 w_6 - w_2 w_5 + w_3 w_4 + \{ (w_1 w_6 - w_2 w_5 - w_3 w_4)^2 - 4w_2 w_3 w_4 w_5 \}^{\frac{1}{2}}, \\ d_2 &= w_1 w_6 + w_2 w_5 - w_3 w_4 + \{ (w_1 w_6 - w_2 w_5 - w_3 w_4)^2 - 4w_2 w_3 w_4 w_5 \}^{\frac{1}{2}}. \end{aligned}$$

The exceptional cases in which the above method will not be permissible are:

(a) When $w_1 w_6 + w_2 w_5 - w_3 w_4 = 0$; (b) when $w_1 w_6 - w_2 w_5 + w_3 w_4 = 0$; (c) when $w_1 w_6 - w_2 w_5 - w_3 w_4 = 0$; (d) when $w_1 = 0$; (e) when $w_6 = 0$.

In case $w_1 w_6 + w_2 w_5 - w_3 w_4 = 0$ it follows that

$$w_2 = w_5 = 0, \quad w_1 w_6 - w_3 w_4 = 0,$$

and the desired inequality is

$$w_1 W_{1, w_4/w_1}.$$

Similarly, $w_1 w_6 - w_2 w_5 + w_3 w_4 = 0$ demands

$$w_3 = w_4 = 0, \quad w_1 w_6 - w_2 w_5 = 0;$$

and the desired inequality is

$$w_6 W_{4, w_5/w_6}.$$

In case $w_1 w_6 - w_2 w_5 - w_3 w_4 = 0$, it is seen by (53) that

$$w_2 = w_5 = 0 \quad \text{or} \quad w_3 = w_4 = 0.$$

In case $w_2 = w_5 = 0$, the desired inequality is

$$w_1 W_{1, w_4/w_1}.$$

In case $w_3 = w_4 = 0$, the desired inequality is

$$w_6 W_{4, w_5/w_6}.$$

When $w_1=0$ or $w_6=0$, we have $w_2=w_3=w_4=w_5=0$, by (49), hence the desired inequality is

$$w_1 W_{1,0} + w_6 W_{4,0}.$$

§ 11. *The Sixteen Bilinear Inequalities (37) Form a Fundamental Set When ξ and η Are Real.*

PROOF. Equations (31) which are valid in this case ($\xi^R; \eta^R$) indicate the coefficients,

$$\left. \begin{aligned} v_1 &= z_{1234} + z_{1243} + z_{2134} + z_{2143}, & v_2 &= z_{1324} + z_{1423} + z_{2413} + z_{2314}, \\ v_3 &= z_{1342} + z_{1432} + z_{2341} + z_{2431}, & v_4 &= z_{4213} + z_{3214} + z_{4123} + z_{3124}, \\ v_5 &= z_{3241} + z_{4231} + z_{4132} + z_{3142}, & v_6 &= z_{3421} + z_{4321} + z_{3412} + z_{4312}, \end{aligned} \right\} \quad (54)$$

Further, let

$$v = v_1 + v_2 + v_3 + v_4 + v_5 + v_6.$$

The general inequality desired is

$$(v_1 J'_{12} J''_{34} + v_2 J'_{13} J''_{24} + v_3 J'_{18} J''_{42} + v_4 J'_{31} J''_{24} + v_5 J'_{31} J''_{42} + v_6 J'_{24} J''_{12}) \xi \bar{\xi} \eta \bar{\eta} \geq 0 \quad (\xi^R; \eta^R; J'; J''). \quad (55)$$

When ξ and η are real, the given inequalities (37) reduce, in accordance with (54), to four inequalities (56).

	v_1	v_2	v_3	v_4	v_5	v_6
$V_{1,u}$	1	0	\bar{u}	u	0	$u\bar{u}$
$V_{2,u}$	1	u	0	0	\bar{u}	$u\bar{u}$
$V_{3,u}$	$u\bar{u}$	0	\bar{u}	u	0	1
$V_{4,u}$	$u\bar{u}$	u	0	0	\bar{u}	1

(56)

Binary operators give necessary and sufficient conditions on v_1, \dots, v_6 .

$$J' = \begin{pmatrix} j'_{11} & j'_{12} \\ j'_{21} & j'_{22} \end{pmatrix}, \quad J'' = \begin{pmatrix} j''_{11} & j''_{12} \\ j''_{21} & j''_{22} \end{pmatrix}$$

give, as an instance of (55),

$$\left. \begin{aligned} &v_1 \{ j'_{11} x_1^2 + (j'_{12} + j'_{21}) x_1 x_2 + j'_{22} x_2^2 \} \\ &\quad \{ j''_{11} y_1^2 + (j''_{12} + j''_{21}) y_1 y_2 + j''_{22} y_2^2 \} \\ &+ v_6 \{ j'_{11} y_1^2 + (j'_{12} + j'_{21}) y_1 y_2 + j'_{22} y_2^2 \} \\ &\quad \{ j''_{11} x_1^2 + (j''_{12} + j''_{21}) x_1 x_2 + j''_{22} x_2^2 \} \\ &+ v_2 \{ j'_{11} x_1 y_1 + j'_{12} x_1 y_2 + j'_{21} x_2 y_1 + j'_{22} x_2 y_2 \} \\ &\quad \{ j''_{11} x_1 y_1 + j''_{12} x_1 y_2 + j''_{21} x_2 y_1 + j''_{22} x_2 y_2 \} \\ &+ v_3 \{ j'_{11} x_1 y_1 + j'_{12} x_1 y_2 + j'_{21} x_2 y_1 + j'_{22} x_2 y_2 \} \\ &\quad \{ j''_{11} x_1 y_1 + j''_{12} x_1 y_2 + j''_{21} x_2 y_1 + j''_{22} x_2 y_2 \} \\ &+ v_4 \{ j'_{11} x_1 y_1 + j'_{12} x_2 y_1 + j'_{21} x_1 y_2 + j'_{22} x_2 y_2 \} \\ &\quad \{ j''_{11} x_1 y_1 + j''_{12} x_2 y_1 + j''_{21} x_1 y_2 + j''_{22} x_2 y_2 \} \\ &+ v_5 \{ j'_{11} x_1 y_1 + j'_{12} x_2 y_1 + j'_{21} x_1 y_2 + j'_{22} x_2 y_2 \} \\ &\quad \{ j''_{11} x_1 y_1 + j''_{12} x_2 y_1 + j''_{21} x_1 y_2 + j''_{22} x_2 y_2 \} \end{aligned} \right\} \geq 0 \quad (57)$$

$$\begin{pmatrix} x_1^R & x_2^R \\ y_1^R & y_2^R \\ j'_{11} \geq 0 & j'_{22} \geq 0 \\ j''_{11} \geq 0 & j''_{22} \geq 0 \\ j'_{21} = \bar{j}'_{12} \\ j''_{21} = \bar{j}''_{12} \\ j'_{11} j'_{22} - j'_{12} j'_{21} \geq 0 \\ j''_{11} j''_{22} - j''_{12} j''_{21} \geq 0 \end{pmatrix}.$$

The cases $(x_1, x_2; y_1, y_2; j'_{11}, j'_{12}, j'_{22}; j''_{11}, j''_{12}, j''_{22}) = (1, 0; 0, 1; 1, 0, 0; 0, 0, 1), (0, 1; 1, 0; 1, 0, 0; 0, 0, 1), (1, 0; 1, 0; 1, 0, 0; 1, 0, 0), (1, 0; 0, 1; 1, i, 1; 1, i, 1), (1, 0; 0, 1; 1, i, 1; 1, -i, 1), (1, 1; 1, -1; 0, 0, 1; 1, 0, 0)$, give as necessary conditions

$$\left. \begin{aligned} v_1 &\geq 0, & v_6 &\geq 0, \\ v_1 + v_2 + v_3 + v_4 + v_5 + v_6 &\geq 0, & v_1 - v_2 + v_3 + v_4 - v_5 + v_6 &\geq 0, \\ v_1 + v_2 - v_3 - v_4 + v_5 + v_6 &\geq 0, & v_1 - v_2 - v_3 - v_4 - v_5 + v_6 &\geq 0, \end{aligned} \right\} \quad (58)$$

whence

$$(v_2 + v_5)^R, \quad (v_3 + v_4)^R.$$

The cases $(0, 1; 1, 0; 1, e', e'^2; 1, e'', e''^2), (1, 0; 0, 1; 1, e', e'^2; 1, -e''i, e''^2), (0, 1; 1, 0; 1, -e'i, e'^2; 1, e'', e''^2), (1, 0; 0, 1; 1, -e'i, e'^2; 1, -e''i, e''^2)$ give the four quadratic forms

$$\left. \begin{aligned} v_1 e'^2 + (v_2 + v_3 + v_4 + v_5) e' e'' + v_6 e''^2 &\geq 0, \\ v_1 e''^2 + (v_3 + v_5 - v_2 - v_4) e' e'' i + v_6 e'^2 &\geq 0, \\ v_1 e'^2 + (v_2 + v_3 - v_4 - v_5) e' e'' i + v_6 e''^2 &\geq 0, \\ v_1 e''^2 + (v_3 + v_4 - v_2 - v_5) e' e'' + v_6 e'^2 &\geq 0, \end{aligned} \right\} \quad (59)$$

whose discriminants give the necessary conditions,

$$\left. \begin{aligned} 4v_1 v_6 - (v_2 + v_3 + v_4 + v_5)^2 &\geq 0, & 4v_1 v_6 + (v_3 + v_5 - v_2 - v_4)^2 &\geq 0, \\ 4v_1 v_6 + (v_2 + v_3 - v_4 - v_5)^2 &\geq 0, & 4v_1 v_6 - (v_3 + v_4 - v_2 - v_5)^2 &\geq 0. \end{aligned} \right\} \quad (60)$$

From (59) it is seen also that $v_2 - v_5$ and $v_3 - v_4$ are pure imaginary, which with (58) gives the conditions,

$$v_2 = \bar{v}_5, \quad v_3 = \bar{v}_4 \quad (61)$$

Using the values

$$(x_1, x_2; y_1, y_2; j'_{11}, j'_{12}, j'_{22}; j''_{11}, j''_{12}, j''_{22}) = (1, 1; -1, 1; 0, 0, 1; 1, -e''i, e''^2),$$

the inequality (57) becomes

$$v e''^2 - 2e''i(v_2 - v_3 + v_4 - v_5) + (v_1 + v_6 - v_2 - v_3 - v_4 - v_5) \geq 0 \quad (e''^R),$$

whence, since the discriminant of this quadratic in e'' must be positive or zero,

$$(v_1 + v_6)^2 - 4(v_2 + v_4)(v_3 + v_5) \geq 0. \quad (62)$$

The values $(1, 1; -1, 1; 1, -e'i, e'^2; 1, -e''i, e''^2)$ give:

$$\left. \begin{aligned} e''^2 \{ v e'^2 - 2(v_2 + v_3 - v_4 - v_5) e' i + (v_1 + v_6 - v_2 - v_3 - v_4 - v_5) \} \\ + 2e'' \{ -2(v_2 - v_3 - v_4 + v_5) e' + (v_2 - v_3 + v_4 - v_5) i (1 - e'^2) \} \\ + \{ (v_1 + v_6 - v_2 - v_3 - v_4 - v_5) e'^2 + 2(v_2 + v_3 - v_4 - v_5) e' i + v \} \end{aligned} \right\} \geq 0 \quad (e'^R, e''^R),$$

whose discriminant may be written as a homogeneous quadratic expression in $(1 - e'^2, 2e')$. Since for e' real, even between the values -1 , and $+1$,

$2e'/(1-e'^2)$ takes every real value, the coefficient of $4e'^2$ and the discriminant of this homogeneous quadratic expression must be positive or zero. Hence,

$$\left. \begin{aligned} (v_1+v_6)^2-4(v_2-v_4)(v_5-v_3) &\geq 0, \\ \{(v_1+v_6)^2-4(v_2v_5+v_3v_4)\}^2-64v_2v_3v_4v_5 &\geq 0. \end{aligned} \right\} \quad (63)$$

From (62) and (63) it follows that

$$\left. \begin{aligned} (v_1+v_6)^2-4(v_2v_5+v_3v_4) &\geq 0, \\ (v_1+v_6)^2-4(v_2v_5-v_3v_4) &\geq 0, \\ (v_1+v_6)^2 &\geq 4v_3v_4, \\ v_1+v_6 &\geq \pm 2(v_3v_4)^{\frac{1}{2}}. \end{aligned} \right\} \quad (64)$$

whence
and
and by (58)

The second condition of (63) can be written

$$\{(v_1+v_6)^2-4(v_2v_5-v_3v_4)\}^2-16v_3v_4(v_1+v_6)^2 \geq 0,$$

and hence by (64),

$$(v_1+v_6)^2-4(v_2v_5-v_3v_4) \geq \pm 4(v_1+v_6)(v_3v_4)^{\frac{1}{2}},$$

and

$$\{v_1+v_6 \pm 2(v_3v_4)^{\frac{1}{2}}\}^2 \geq 4v_2v_5,$$

giving, by (64),

$$v_1+v_6 \pm 2(v_3v_4)^{\frac{1}{2}} \geq \pm 2(v_2v_5)^{\frac{1}{2}}. \quad (65)$$

In building the general inequality (55) as the sum of positive or zero multiples of the fundamental inequalities (56), it is desirable first to build those for which $v_1=v_6=\frac{1}{2}(v_1+v_6) \neq 0$. For such, aside from exceptional cases, the general inequality has the form

$$\frac{1}{2}(v_3v_4)^{\frac{1}{2}}(V_{1, (v_4/v_3)^{\frac{1}{2}}} + V_{3, (v_4/v_3)^{\frac{1}{2}}}) + \frac{1}{4}d(V_{2, 2v_2/d} + V_{4, 2v_2/d}),$$

where

$$d = v_1+v_6-2(v_3v_4)^{\frac{1}{2}} + [\{v_1+v_6-2(v_3v_4)^{\frac{1}{2}}\}^2-4v_2v_5]^{\frac{1}{2}}.$$

When $v_1+v_6-2(v_3v_4)^{\frac{1}{2}}=0$, it follows from (65) that $v_2=v_5=0$, hence the desired inequality may be expressed as

$$\frac{1}{4}(v_1+v_6)(V_{1, 2v_4/(v_1+v_6)} + V_{3, 2v_4/(v_1+v_6)}).$$

In case $v_1 \neq v_6$, the desired inequality is secured by adding to that already built

$$(v_1-v_6)V_{1,0} \text{ or } (v_6-v_1)V_{3,0},$$

according as v_1 is greater than or less than v_6 .

If $v_1=0$, or $v_6=0$, then, by (60), $v_2=v_3=v_4=v_5=0$, hence the desired inequality has the form

$$v_1V_{1,0} + v_6V_{3,0}.$$

All portions of this paper which refer to ξ^R, η^R ; namely, § 4, § 8, and § 11, are valid for ξ and η pure imaginary, as an inequality for ξ and η pure imaginary reduces at once to the same inequality for ξ and η real.

On the three corresponding larger problems, retaining general operator and general operand, the writer has made considerable progress. In each case the nature of the coefficients a_{ijkl} and z_{ijkl} has been determined, and many necessary conditions on the coefficients have been secured, by use of binary, ternary, quaternary and quinary operators. As to the nature of the coefficients it has been found that in the problems corresponding to those of Part I and Part II,

$$\begin{array}{cccc} a_{1234} \geq 0, & a_{1243} \geq 0, & a_{3421} \geq 0, & a_{4321} \geq 0, \\ a_{1342}^R, & a_{1432}^R, & a_{2341}^R, & a_{2431}^R, \\ a_{1324} = \overline{a_{4231}}, & a_{1423} = \overline{a_{3241}}, & & \end{array}$$

while in the problem on bilinear forms,

$$\begin{array}{cccc} z_{1234} \geq 0, & z_{1243} \geq 0, & z_{3421} \geq 0, & z_{4321} \geq 0, \\ z_{3412} \geq 0, & z_{4312} \geq 0, & z_{2134} \geq 0, & z_{2143} \geq 0, \\ z_{1324} = \overline{z_{4231}}, & z_{1342} = \overline{z_{4213}}, & z_{1423} = \overline{z_{3241}}, & z_{1432} = \overline{z_{3214}}, \\ z_{2341} = \overline{z_{4123}}, & z_{2431} = \overline{z_{3124}}, & z_{2413} = \overline{z_{3142}}, & z_{2314} = \overline{z_{4132}}. \end{array}$$

A Trigonometrical Sum and the Gibbs' Phenomenon in Fourier's Series.*

BY H. S. CARSLAW.

§ 1. If $f(x)$ is a function which can be expanded in a Fourier's Series in the interval $0 \leq x \leq 2\pi$, with a discontinuity at $x=a$ such that $f(a-0)$ and $f(a+0)$ exist, the sum of the Fourier's Series for $x=a$ is $\frac{1}{2}\{f(a+0) + f(a-0)\}$.

Denoting by $S_n(x)$ the sum of the terms up to and including those in $\sin nx$ and $\cos nx$, the curves $y=S_n(x)$ are usually spoken of as the approximation curves for the series. Up till 1899 it was believed that each approximation curve, for large values of n , passes at a steep gradient from a point near $(a, f(a-0))$ to a point near $(a, \frac{1}{2}\{f(a+0) + f(a-0)\})$, and then on to a point near $(a, f(a+0))$, afterwards oscillating about the curve $y=f(x)$ till another discontinuity of $f(x)$ is met.

In 1899 J. Willard Gibbs pointed out† that the approximation curves for the sine series

$$2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots)$$

do not behave in this way.

In the interval $0 \leq x \leq 2\pi$, the sum of this series is $f(x)$, where $f(x)$ is defined as follows:

$$\left. \begin{aligned} f(0) &= f(2\pi) = 0, \\ f(x) &= x, \quad 0 < x < \pi, \\ f(x) &= x - 2\pi, \quad \pi < x < 2\pi. \end{aligned} \right\}$$

Gibbs stated in effect that the curve $y=S_n(x)$ for this series, for large values of n , rises just before $x=\pi$ to a point very nearly at a height $2 \int_0^\pi \frac{\sin x}{x} dx$ above the axis of x , passes from that point through $(\pi, 0)$ at a steep gradient to a point very nearly at the same distance $2 \int_0^\pi \frac{\sin x}{x} dx$ below the axis, while in the rest of the interval $0 \leq x \leq 2\pi$ it oscillates above and below the curve $y=f(x)$.

* Communicated to the American Mathematical Society, October, 1915.

† *Nature*, Vol. LIX (1899), p. 606.

His statement was not accompanied by any proof. Though the remainder of the correspondence, of which his letter formed a part, attracted no little attention, this remarkable observation remained practically unnoticed for several years. In 1906 Bôcher returned to the subject in a memoir on Fourier's Series,* and extended Gibbs' results in two directions.

First of all he discussed the series

$$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots,$$

which, in the interval $0 \leq x \leq 2\pi$, represents the function $f(x)$ defined as follows:

$$\left. \begin{aligned} f(0) &= f(2\pi) = 0, \\ f(x) &= \frac{1}{2}(\pi - x) \dots 0 < x < 2\pi. \end{aligned} \right\}$$

He showed that the approximation curve

$$y = \sin x + \frac{1}{2} \sin 2x + \dots + \frac{1}{n} \sin nx = S_n(x),$$

in the interval $0 \leq x \leq 2\pi$, and for large values of n , takes the form of a wavy curve which keeps crossing and recrossing the line $y = \frac{1}{2}(\pi - x)$ and reaches its greatest distance from that line (distances being measured parallel to the axis of y) at the points x_1, x_2, \dots, x_{2n} , where $x_r = \frac{2r\pi}{2n+1}$.

Further, these greatest distances in the r -th wave are, for large values of n , approximately equal to their limiting values

$$\frac{1}{2}\pi - \int_0^\pi \frac{\sin x}{x} dx.$$

In the second place he showed that the phenomenon in question also appears in the Fourier's Series for a large class of functions. How large this class is will appear in the concluding section of this paper.

In Bôcher's own words:† If $S_n(x)$ denotes the sum of the Fourier's expansion of $f(x)$, the curve $y = S_n(x)$ will, for large values of n , pass in almost a vertical direction through a point whose abscissa is a and whose ordinate is almost $\frac{1}{2}\{f(a+0) + f(a-0)\}$. The curve then rises and falls abruptly on the two sides of this point to the neighborhood of the curve

* *Annals of Mathematics* (2), Vol. VII (1906). See also a recent paper in *Crelle's Journal*, Bd. 144 (1914), entitled "On Gibbs' Phenomenon."

Reference should also be made to Runge's "Theorie und Praxis der Reihen" (1904), pp. 170-180. A certain series is there discussed, and the nature of the jump in the approximation curve described. But no reference is made to Gibbs, and the example seems to have been regarded as quite an isolated one.

† *Loc. cit.*, *Annals of Mathematics*, p. 131. Both Theorems I and II, p. 131, should be noted.

$y=f(x)$, and oscillates about this curve, lying alternately above and below it. The highest (or lowest) point of the k -th wave to the right and left of a will, for large values of n , be approximately at the points

$$a \pm \frac{2k\pi}{2n+1},$$

and the height of these waves will be approximately

$$\frac{f(a+0)-f(a-0)}{\pi} P_k,$$

(where $P_k = \frac{1}{2}\pi - \int_0^{k\pi} \frac{\sin x}{x} dx$).

In the above sentence, a is a value of x at which $f(x)$ has an ordinary discontinuity, and Bôcher adds a footnote to the effect that "it must be borne in mind that we measure the height of a wave from the curve $y=f(x)$ in a direction parallel to the axis of y ."

§ 2. In the function studied by Bôcher, $\frac{1}{2}(\pi-x)$,

$$\begin{aligned} S_n(x) &= \sin x + \frac{1}{2} \sin 2x + \dots + \frac{1}{n} \sin nx \\ &= \int_0^x (\cos \alpha + \cos 2\alpha + \dots + \cos n\alpha) d\alpha \\ &= \frac{1}{2} \int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha - \frac{1}{2}x. \end{aligned}$$

The properties of the maxima and minima of $S_n(x)$ are not so easy to obtain, nor are they so useful in this case as those of

$$\begin{aligned} R_n(x) &= \frac{1}{2}(\pi-x) - S_n(x) \\ &= \frac{1}{2}\pi - \frac{1}{2} \int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha. \end{aligned}$$

It is the maxima and minima of $R_n(x)$ with which Bôcher deals in his memoir.

Gronwall has discussed* the somewhat complicated properties of the maxima and minima of $S_n(x)$ for this series, and deduced Gibbs' Phenomenon for the first wave, and the general case of Fourier's Series.

In this paper I use the series

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots),$$

* *Mathematische Annalen*, Bd. LXXII (1912). Some of Gronwall's results had been published a few months earlier by Dunham Jackson in a paper in the *Rendiconti di Circolo Matematico di Palermo*, T. XXXII (1911).

which, in the interval $-\pi \leq x \leq \pi$,* represents the function $f(x)$ defined by the equations

$$\left. \begin{aligned} f(-\pi) &= f(0) = f(\pi) = 0, \\ f(x) &= -\frac{1}{2}\pi \dots -\pi < x < 0, \\ f(x) &= \frac{1}{2}\pi \dots 0 < x < \pi. \end{aligned} \right\}$$

I take $S_n(x)$ for the sum of the first n terms of this series, so that

$$S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right).$$

A number of interesting properties of the turning points of $y = S_n(x)$ are obtained by quite simple methods; and all the features of Gibbs' Phenomenon for this series follow immediately from these properties.

In the concluding article the extension to the general case of Fourier's Series is given; but the method does not differ materially from that of Bôcher.

It is, of course, not an unusual thing that the curves $y = S_n(x)$ for a series, whose terms are continuous functions, differ considerably, even for large values of n , from the curve $y = \lim_{n \rightarrow \infty} (S_n(x)) = S(x)$.

They certainly do so in the neighborhood of a point where $S(x)$, the sum of the series, is discontinuous. And they also may do so in the neighborhood of a point where $S(x)$ is continuous.†

Ex. (i). If

$$S_n(x) = \frac{nx}{1+n^2x^2}, \quad x \geq 0,$$

we have

$$S(x) = 0, \quad x \geq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}$ at $x = \frac{1}{n}$. As n gets larger and larger, this summit is pushed towards $x = 0$ but its height remains equal to $\frac{1}{2}$.

Ex. (ii). If

$$S_n(x) = \frac{n^2x}{1+n^3x^2}, \quad x \geq 0,$$

we have

$$S(x) = 0, \quad x \geq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}n^{\frac{1}{3}}$ at $x = \frac{1}{n^{\frac{1}{3}}}$. As n gets larger and larger, this summit is pushed towards $x = 0$ but its height increases without limit.

* It is more convenient for my purpose to take the interval $-\pi \leq x \leq \pi$ than $0 \leq x \leq 2\pi$. This does not affect the argument to any extent.

† Cf. Osgood, *Bulletin of the American Math. Soc.* (2), Vol. III (1897), p. 63; AMERICAN JOURNAL OF MATHEMATICS, Vol. XIX (1897), p. 155; Hobson, "Theory of Functions of a Real Variable," p. 480; Bromwich, "Theory of Infinite Series," p. 110; Carslaw, "Fourier's Series and Integrals," Ch. III.

The existence of maxima (or minima) of $S_n(x)$, the abscissæ of which tend towards a (a being a value of x for which the sum of the series is discontinuous), while their ordinates remain at a finite distance from $S(a+0)$ and $S(a-0)$, as n increases, is the chief feature of the Gibbs' Phenomenon in Fourier's Series. And it is most remarkable that its occurrence in Fourier's Series remained undiscovered till so recent a date.*

THE TRIGONOMETRICAL SUM

$$S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right).$$

§ 3. If we define $f(x)$ by the equations

$$\left. \begin{aligned} f(-\pi) &= f(0) = f(\pi), \\ f(x) &= \frac{1}{2}\pi \dots 0 < x < \pi, \\ f(x) &= -\frac{1}{2}\pi \dots -\pi < x < 0, \end{aligned} \right\}$$

the Fourier's Series for $f(x)$ in the interval $-\pi \leq x \leq \pi$ is

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

We denote by $S_n(x)$ this sum up to and including the term in $\sin(2n-1)x$.

Then

$$S_n(x) = 2 \int_0^x (\cos x + \cos 3x + \dots + \cos (2n-1)x) dx = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha.$$

Since $S_n(x)$ is an *odd* function, we need only consider the interval $0 \leq x \leq \pi$.

We proceed to obtain the properties of the maxima and minima of this function $S_n(x)$.

I. Since, for any integer m , $\sin(2m-1)(\frac{1}{2}\pi + x') = \sin(2m-1)(\frac{1}{2}\pi - x')$, it follows from the series that $S_n(x)$ is *symmetrical* about $x = \frac{1}{2}\pi$, and when $x=0$ and $x=\pi$ it is zero.

II. When $0 < x < \pi$, $S_n(x)$ is *positive*.

From (I) we need only consider $0 < x \leq \frac{1}{2}\pi$. We have

$$S_n(x) = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha = \frac{1}{2n} \int_0^{2nx} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha, \quad 0 < x \leq \frac{1}{2}\pi.$$

The denominator in the integrand is positive and continually increases in the

* Cf. also Weyl: (i) "Die Gibbs'sche Erscheinung in der Theorie der Kugel-Funktionen," *Rend. Circ. Mat. d. Palermo*, T. XXIX (1910); (ii) "Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene," *ibid.*, T. XXX (1910).

interval of integration. By considering the successive waves in the graph of $\sin \alpha \operatorname{cosec} \frac{\alpha}{2n}$, the last of which may or may not be completed, it is clear that the integral is positive.

III. *The turning points of $y=S_n(x)$ are given by*

$$\begin{cases} x_1 = \frac{\pi}{2n}, x_3 = \frac{3\pi}{2n}, \dots, x_{2n-1} = \frac{2n-1}{2n}\pi \text{ (maxima),} \\ x_2 = \frac{\pi}{n}, x_4 = \frac{2\pi}{n}, \dots, x_{2(n-1)} = \frac{n-1}{n}\pi \text{ (minima).} \end{cases}$$

We have

$$y = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha, \text{ and } \frac{dy}{dx} = \frac{\sin 2nx}{\sin x}.$$

The result follows at once.

IV. *As we proceed from $x=0$ to $x=\frac{1}{2}\pi$, the heights of the maxima continually diminish, and the heights of the minima continually increase, n being kept fixed.*

Consider two consecutive maxima in the interval $0 < x \leq \frac{1}{2}\pi$, namely, $S_n\left(\frac{2m-1}{2n}\pi\right)$ and $S_n\left(\frac{2m+1}{2n}\pi\right)$, m being a positive integer less than or equal to $\frac{1}{2}(n-1)$. We have

$$\begin{aligned} S_n\left(\frac{2m-1}{2n}\pi\right) - S_n\left(\frac{2m+1}{2n}\pi\right) &= \frac{1}{2n} \int_{(2m+1)\pi}^{(2m-1)\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha \\ &= -\frac{1}{2n} \left\{ \int_{(2m-1)\pi}^{2m\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha + \int_{2m\pi}^{(2m+1)\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha \right\}. \end{aligned}$$

The denominator in both integrands is positive and it continually increases in the interval $(2m-1)\pi \leq \alpha \leq (2m+1)\pi$; also the numerator in the first is continually negative and in the second continually positive; the absolute values for elements at equal distances from $(2m-1)\pi$ and $2m\pi$ being the same.

Thus the result follows. Similarly for the minima, we have to examine the sign of

$$S_n\left(\frac{m-1}{n}\pi\right) - S_n\left(\frac{m}{n}\pi\right),$$

where m is a positive integer less than or equal to $\frac{1}{2}n$.

V. The first maximum to the right of $x=0$ is at $x=\frac{\pi}{2n}$ and its height continually diminishes as n increases. When n tends to infinity, its limit is

$$\int_0^{\pi} \frac{\sin x}{x} dx.$$

We have

$$S_n\left(\frac{\pi}{2n}\right) = \int_0^{\pi} \frac{\sin 2n\alpha}{\sin \alpha} d\alpha = \frac{1}{2n} \int_0^{\pi} \sin \alpha \operatorname{cosec} \frac{\alpha}{2n} d\alpha,$$

and

$$S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2n+2}\right) = \int_0^{\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha.$$

Since $\alpha/\sin \alpha$ continually increases from 1 to ∞ , as α passes from 0 to π , it is clear that in the interval with which we have to deal

$$\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} > 0.$$

Thus

$$S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2(n+1)}\right) > 0.$$

But, from (I), $S_n(x)$ is positive when $0 < x < \pi$.

It follows that $S_n\left(\frac{\pi}{2n}\right)$ tends to a limit as n tends to infinity.

The value of this limit can be obtained by the method used by Bôcher for the integral $\int_0^{\pi} \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha$.^{*} But it is readily obtained from the definition of the Definite Integral as the limit of a sum.

For we have

$$\begin{aligned} S_n\left(\frac{\pi}{2n}\right) &= 2\left(\frac{\pi}{2n}\right) \left\{ \frac{2n}{\pi} \sin \frac{\pi}{2n} + \frac{2n}{3\pi} \sin \frac{3\pi}{2n} + \dots + \frac{2n}{(2n-1)\pi} \sin \frac{2n-1}{2n} \pi \right\} \\ &= 2 \sum_{\substack{m=1 \\ 2nh=\pi}}^{2n} \left(\frac{\sin mh}{mh} h \right) - \sum_{\substack{m=1 \\ nh=\pi}}^n \left(\frac{\sin mh}{mh} h \right). \end{aligned}$$

Therefore

$$\text{Lt.}_{n \rightarrow \infty} S_n\left(\frac{\pi}{2n}\right) = 2 \int_0^{\pi} \frac{\sin x}{x} dx - \int_0^{\pi} \frac{\sin x}{x} dx = \int_0^{\pi} \frac{\sin x}{x} dx.$$

^{*} Loc. cit., *Annals of Mathematics*, p. 124. Also Hobson, loc. cit., p. 649.

VI. The result obtained in (V) for the first wave is a special case of the following:

The r -th maximum to the right of $x=0$ is at $x_{2r-1} = \frac{2r-1}{2n}\pi$, and its height continually diminishes as n increases, r being kept constant. When n tends to infinity, its limit is $\int_0^{(2r-1)\pi} \frac{\sin x}{x} dx$, which is greater than $\frac{1}{2}\pi$.

The r -th minimum to the right of $x=0$ is at $x_{2r} = \frac{r}{n}\pi$, and its height continually increases as n increases, r being kept constant. When n tends to infinity, its limit is $\int_0^{2r\pi} \frac{\sin x}{x} dx$, which is less than $\frac{1}{2}\pi$.

To prove these theorems we consider first the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha,$$

m being a positive integer less than or equal to $2n-1$, so that $0 < \frac{\alpha}{2n} < \pi$ in the interval of integration.

Then

$$F(\alpha) = \frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} > 0$$

in this interval. (Cf. (V)).

Further,

$$\begin{aligned} F'(\alpha) &= \frac{1}{(2n+2)^2} \cos \frac{\alpha}{2n+2} \operatorname{cosec}^2 \frac{\alpha}{2n+2} - \frac{1}{(2n)^2} \cos \frac{\alpha}{2n} \operatorname{cosec}^2 \frac{\alpha}{2n} \\ &= \alpha^{-2} \{ \phi^2 \cos \phi \operatorname{cosec}^2 \phi - \psi^2 \cos \psi \operatorname{cosec}^2 \psi \}, \end{aligned}$$

where $\phi = \alpha/(2n+2)$ and $\psi = \alpha/2n$.

But

$$\frac{d}{d\phi} (\phi^2 \cos \phi \operatorname{cosec}^2 \phi) = -\phi \operatorname{cosec}^3 \phi [\phi(1 \mp \cos \phi)^2 \pm 2 \cos \phi (\phi \mp \sin \phi)].$$

And the right-hand side of the equation will be seen to be negative, choosing the upper signs for $0 < \phi < \frac{1}{2}\pi$ and the lower for $\frac{1}{2}\pi < \phi < \pi$.

Therefore $\phi^2 \cos \phi \operatorname{cosec}^2 \phi$ diminishes as ϕ increases from 0 to π .

It follows, from the expression for $F'(\alpha)$, that $F'(\alpha) > 0$, and $F(\alpha)$ increases with α in the interval of integration.

The curve

$$y = \sin x \left(\frac{1}{2n} \operatorname{cosec} \frac{x}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{x}{2n+2} \right), \dots, 0 < x < m\pi,$$

thus consists of a succession of waves of length π , alternately above and below the axis, and the absolute values of the ordinates at points at the same distance from the beginning of each wave continually increase.

It follows that, when m is equal to $2, 4, \dots, 2(n-1)$, the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha$$

is negative; and, when m is equal to $1, 3, \dots, 2n-1$, this integral is positive.

Returning to the maxima and minima, we have, for the r -th maximum to the right of $x=0$,

$$\begin{aligned} S_n(x_{2r-1}) - S_{n+1}(x_{2r-1}) &= \int_0^{\frac{2r-1}{2n}\pi} \frac{\sin 2n\alpha}{\sin \alpha} d\alpha - \int_0^{\frac{2r-1}{2(n+1)}\pi} \frac{\sin 2(n+1)\alpha}{\sin \alpha} d\alpha \\ &= \int_0^{(2r-1)\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha. \end{aligned}$$

Therefore from the above argument, $S_n(x_{2r-1}) > S_{n+1}(x_{2r-1})$.

Also for the r -th minimum to the right of $x=0$, we have

$$S_n(x_{2r}) - S_{n+1}(x_{2r}) = \int_0^{2r\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha,$$

and $S_n(x_{2r}) < S_{n+1}(x_{2r})$.

By an argument similar to that at the close of (V) we have

$$\operatorname{Lt.}_{n \rightarrow \infty} S_n(x_r) = \int_0^{r\pi} \frac{\sin x}{x} dx. *$$

It is clear that these limiting values are all greater than $\frac{1}{2}\pi$ for the maxima, and positive and less than $\frac{1}{2}\pi$ for the minima.

THE GIBBS' PHENOMENON FOR THE SERIES

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

§ 4. From the Theorems (I)-(VI) of § 3 all the features of the Gibbs' Phenomenon for the series

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots) \dots -\pi \leq x \leq \pi$$

follow immediately.

* For the values of $\int_0^{r\pi} \frac{\sin x}{x} dx$, see Bôcher, *loc. cit.*, *Annals of Mathematics*, p. 129.

It is obvious that we need only examine the interval $0 \leq x \leq \pi$, and that the discontinuity occurs at $x=0$.

For large values of n , the curve

$$y = S_n(x),$$

where $S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin(2n-1)x \right)$, rises at a steep gradient from the origin to its first maximum, which is very near, but above, the point $\left(0, \int_0^\pi \frac{\sin x}{x} dx\right)$ § 3, V. The curve, then, falls at a steep gradient, without reaching the axis of x § 3, II, to its first minimum, which is very near, but below, the point $\left(0, \int_0^{2\pi} \frac{\sin x}{x} dx\right)$ § 3, VI. It then oscillates above and below the line $y = \frac{1}{2}\pi$, the heights (and depths) of the waves continually diminishing as we proceed from $x=0$ to $x = \frac{1}{2}\pi$ § 3, IV; and from $x = \frac{1}{2}\pi$ to $x = \pi$, the procedure is reversed, the curve in the interval $0 \leq x \leq \pi$ being symmetrical about $x = \frac{1}{2}\pi$ § 3, I.

The highest (or lowest) point of the r -th wave to the right of $x=0$ will, for large values of n , be at a point whose abscissa is $\frac{r\pi}{2n}$ § 3, III and whose ordinate is very nearly $\int_0^{r\pi} \frac{\sin x}{x} dx$ § 3, VI.

By increasing n the curve for $0 \leq x \leq \pi$ can be brought as close as we please to the lines

$$\left. \begin{array}{ll} x=0, & 0 < y < \int_0^\pi \frac{\sin x}{x} dx, \\ 0 < x < \pi, & y = \frac{\pi}{2}, \\ x=\pi, & 0 < y < \int_0^\pi \frac{\sin x}{x} dx. \end{array} \right\}$$

We may state these results more definitely as follows:

(i) If ε is any positive number, as small as we please, there is a positive integer ν' such that

$$|\tfrac{1}{2}\pi - S_n(x)| < \varepsilon \text{ for } n \geq \nu', \varepsilon \leq x \leq \tfrac{1}{2}\pi.$$

This follows from the uniform convergence of the Fourier's Series for $f(x)$ in an interval which does not include a discontinuity of $f(x)$.

(ii) Since the height of the first maximum to the right of $x=0$ tends

from above to $\int_0^\pi \frac{\sin x}{x} dx$ as n tends to infinity, there is a positive integer ν'' such that

$$0 < S_n\left(\frac{\pi}{2n}\right) - \int_0^\pi \frac{\sin x}{x} dx < \varepsilon \text{ for } n \geq \nu''.$$

(iii) Let ν''' be the integer next greater than $\frac{\pi}{2\varepsilon}$. Then the abscissa of the first maximum to the right of $x=0$, for $n \geq \nu'''$, is less than ε .

It follows from (i), (ii) and (iii) that, if ν is the greatest of the positive integers ν' , ν'' and ν''' , the curve $y=S_n(x)$, for $n \geq \nu$, behaves as follows:

It rises at a steep gradient from the origin to its first maximum, which is above $\int_0^\pi \frac{\sin x}{x} dx$ and within the rectangle

$$0 < x < \varepsilon, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \varepsilon.$$

After leaving this rectangle, in which there may be many oscillations about $y = \frac{1}{2}\pi$, it remains within the rectangle $\varepsilon < x < \pi - \varepsilon$, $\frac{1}{2}\pi - \varepsilon < y < \frac{1}{2}\pi + \varepsilon$.

Finally, it enters the rectangle

$$\pi - \varepsilon < x < \pi, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \varepsilon,$$

and the procedure in the first region is repeated. *

THE GIBBS' PHENOMENON FOR THE FOURIER'S SERIES IN GENERAL.

§ 5. Let $f(x)$ be a function with an ordinary discontinuity when $x=a$, which can be expanded in a Fourier's Series in the interval $-\pi \leq x \leq \pi$.

Denote as usual by $f(a+0)$ and $f(a-0)$, the values towards which $f(x)$ tends as x approaches a from above or below. It will be convenient to consider $f(a+0)$ as greater than $f(a-0)$ in the description of the curve, but this restriction is in no way necessary.

Let $\phi(x-a) = 2 \sum_{r=1}^{\infty} \frac{1}{(2r-1)} \sin(2r-1)(x-a)$. Then

$$\left. \begin{aligned} \phi(x-a) &= \frac{1}{2}\pi, \text{ when } a < x < \pi+a, \\ \phi(x-a) &= -\frac{1}{2}\pi, \text{ when } -\pi+a < x < a, \\ \phi(+0) &= \frac{1}{2}\pi, \quad \phi(-0) = -\frac{1}{2}\pi, \\ \phi(0) &= 0 \text{ and } \phi(x) = \phi(x+2\pi). \end{aligned} \right\}$$

* The cosine series

$$\frac{\pi}{4} - \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x + \dots \right]$$

which represents 0 in the interval $0 \leq x < \frac{\pi}{2}$ and $\frac{\pi}{2}$ in the interval $\frac{\pi}{2} < x \leq \pi$ can be treated in the same way as the series discussed in this article.

Now put

$$\psi(x) = f(x) - \frac{1}{2}\{f(a+0) + f(a-0)\} - \frac{1}{\pi}\{f(a+0) - f(a-0)\}\phi(x-a),$$

and let $f(x)$, for $x=a$, be defined as $\frac{1}{2}\{f(a+0) + f(a-0)\}$.

Then $\psi(a+0) = \psi(a-0) = \psi(a) = 0$, and $\psi(x)$ is continuous at $x=a$.

The following distinct steps in the argument are numbered for the sake of clearness:

(i) Since $\psi(x)$ is continuous at $x=a$ and $\psi(a)=0$, if ϵ is a positive number, as small as we please, a number δ exists such that

$$|\psi(x)| < \frac{\epsilon}{4} \text{ for } |x-a| < \delta.$$

If δ is not originally less than ϵ , we can choose this part of δ for our interval.

(ii) $\psi(x)$ can be expanded in a Fourier's Series, this series being uniformly convergent in an interval $\alpha \leq x \leq \beta$ which includes no other discontinuity of $f(x)$ and $\phi(x-a)$ than $x=a$.

Let $s_n(x)$, $\phi_n(x-a)$ and $\sigma_n(x)$ be the sums of the terms up to and including those in $\sin nx$ and $\cos nx$ in the Fourier's Series for $f(x)$, $\phi(x-a)$ and $\psi(x)$. Then ϵ being the positive number of (i), as small as we please, there exists a positive integer ν' such that

$$|\sigma_n(x) - \psi(x)| < \frac{\epsilon}{4} \text{ for } n \leq \nu' \text{ in } \alpha \leq x \leq \beta.$$

Also

$$|\sigma_n(x)| \leq |\sigma_n(x) - \psi(x)| + |\psi(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{4} < \frac{\epsilon}{2}$$

in $|x-a| < \delta$, if $\alpha < a - \delta < a < a + \delta < \beta$.

(iii) Now if n is even, the first maximum in $\phi_n(x-a)$ to the right of $x=a$ is at $a + \frac{\pi}{n}$; and if n is odd, it is at $a + \frac{\pi}{n+1}$. In either case there exists a positive integer ν'' such that the height of the first maximum lies between $\int_0^\pi \frac{\sin x}{x} dx$ and $\int_0^\pi \frac{\sin x}{x} dx + \frac{\pi\epsilon}{2\{f(a+0) - f(a-0)\}}$ for $n \geq \nu''$.

(iv) This first maximum will have its abscissa between a and $a + \delta$, provided that $\frac{\pi}{n} < \delta$.

Let ν''' be the first positive integer which satisfies this inequality.

(v) In the interval $a + \delta \leq x \leq \beta$, $s_n(x)$ converges uniformly to $f(x)$. Therefore, there exists a positive integer $\nu^{(iv)}$ such that for $n \geq \nu^{(iv)}$

$$|f(x) - s_n(x)| < \varepsilon \text{ in this interval.}$$

Now from the equation defining $\psi(x)$ we have

$$s_n(x) = \frac{1}{2}(f(a+0) + f(a-0)) + \frac{f(a+0) - f(a-0)}{\pi} \phi_n(x-a) + \sigma_n(x).$$

It follows from (i)-(v) that if ν is the first positive integer greater than ν', ν'', ν''' and $\nu^{(iv)}$, the curve $y = s_n(x)$ in the interval $a \leq x \leq \beta$, behaves as follows:

When $x = a$, it passes through a point whose ordinate is within $\frac{\varepsilon}{2}$ of $\frac{1}{2}(f(a+0) + f(a-0))$, and ascends at a steep gradient to its first maximum, which is at a height within ε of

$$\frac{1}{2}\{f(a+0) + f(a-0)\} + \frac{f(a+0) - f(a-0)}{\pi} \int_0^\pi \frac{\sin x}{x} dx.$$

This may be written

$$f(a+0) - \frac{f(a+0) - f(a-0)}{\pi} \int_\pi^\infty \frac{\sin x}{x} dx,$$

and, from Bôcher's table, referred to in § 3, we have

$$\int_\pi^\infty \frac{\sin x}{x} dx = -0.2811.$$

It then oscillates about $y = f(x)$ from $x = a$ to $a + \delta$, the character of the waves being determined by the function $\phi_n(x-a)$, since the term $\sigma_n(x)$ only adds a quantity less than $\frac{\varepsilon}{2}$ to the ordinate.

And on passing beyond $x = a + \delta$, the curve enters and remains within, the strip of width 2ε enclosing $y = f(x)$ from $x = a + \delta$ to $x = \beta$.

On the other side of the point a a similar set of circumstances can be established.

APPENDIX.

The following diagrams are the approximation curves

$$y = S_n(x)$$

of § 3 above, for $n=1, 2, 3$ and 4 . They illustrate the results (I-VI) of that section.

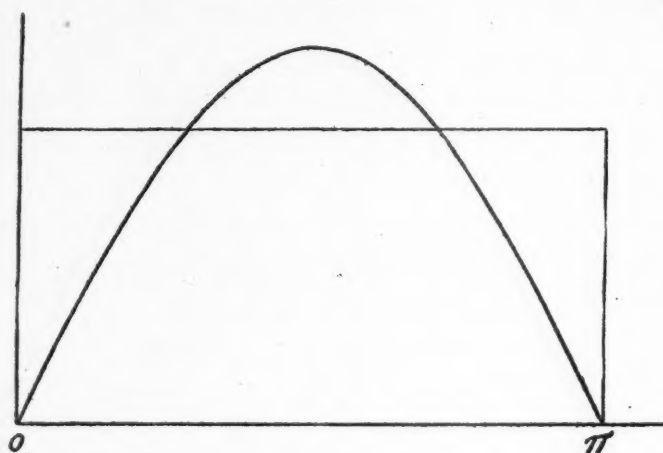


FIG. 1.

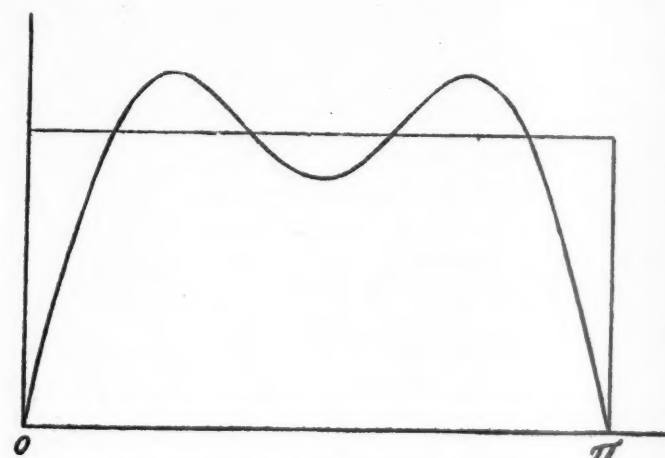


FIG. 2.

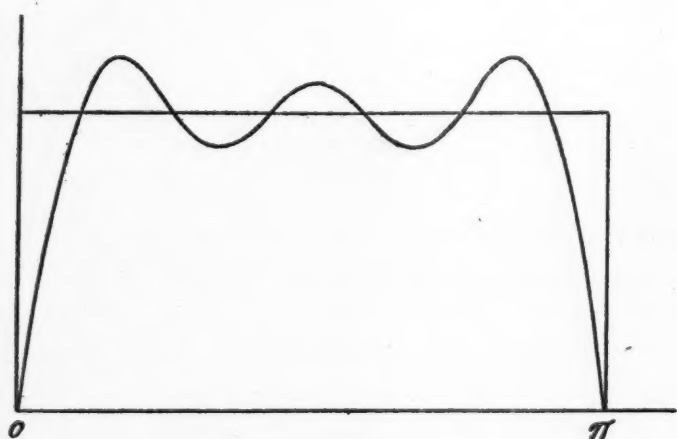


FIG. 3.

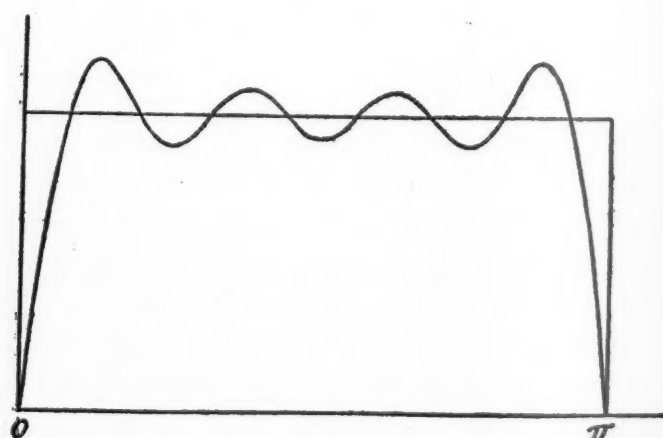


FIG. 4.

SYDNEY, AUSTRALIA, August, 1915.

On the Relation between Some Important Notions of Projective and Metrical Differential Geometry.*

BY F. M. MORRISON.

I. INTRODUCTION.

In his papers on the "Projective Differential Geometry of Curved Surfaces"† Professor E. J. Wilczynski has shown that the projective differential geometry of a surface may be based upon the consideration of a completely integrable system of two linear partial differential equations of the second order, which may be reduced to what he calls the intermediate form

$$y_{uu} + 2ay_u + 2by_v + cy = 0, \quad y_{vv} + 2a'y_u + 2b'y_v + c'y = 0. \quad (1)$$

Such a system of equations has just four linearly independent solutions, $y', y'', y^{(3)}, y^{(4)}$. If these be taken as the homogeneous coordinates of a point P_y in space, as u and v assume all of their values, the point P_y will describe a surface S , an integral surface of equations (1). This surface will be non-degenerate and non-developable, and will have the curves $u = \text{const.}$ and $v = \text{const.}$ as asymptotic lines. The most general integral surface of equations (1) is a projective transformation of any particular one, so that the coefficients of equations (1), in so far as they are intrinsically connected with the surface at all, will involve only its projective properties. But these coefficients also depend upon certain accidental elements of the analytic representation of the surface. To eliminate these accidental elements the notions of invariants and covariants are introduced.‡ The projective properties of the surface are then expressed by invariant equations or system of equations. Other surfaces and other geometrical configurations which have a projective relation to it will be given by the covariants.

The four fundamental semi-covariants of system (1) are

$$y, \quad z = y_u + ay, \quad \rho = y_v + b'y, \quad \sigma = y_{uv} + b'y_u + ay_v + \frac{1}{2}(a_v + b'_u + 2ab')y. \quad (2)$$

* Presented to the American Mathematical Society, San Francisco Section, Seattle, Wash., May, 1914.

† *Transactions of the American Mathematical Society*, Vols. VIII, IX and X.

‡ *Transactions*, Vol. VIII, p. 241, et seq.

If four linearly independent solutions $y', y'', y^{(3)}, y^{(4)}$ of (1) be put for y in the semi-covariants (2), three points P_z, P_ρ, P_σ will be obtained which are semi-covariantly connected with the point P_y .

By a properly chosen transformation of the form $y = \lambda \bar{y}$ equations (1) may be changed into the canonical form

$$y_{uu} + 2by_v + fy = 0, \quad y_{vv} + 2a'y_u + gy = 0. \quad (3)$$

The semi-covariants of (1) and (3) are connected with each other by the equations $y = \lambda \bar{y}, z = \lambda \bar{z}, \rho = \lambda \bar{\rho}, \sigma = \lambda \bar{\sigma}$. The semi-covariant points are the same for both (1) and (3). These points $P_y, P_z, P_\rho, P_\sigma$ form in general a non-degenerate tetrahedron which Professor Wilczynski has systematically used as tetrahedron of reference, for the purpose of studying the properties of a surface in the neighborhood of any one of its points.*

If the Cartesian coordinates of the surface are regarded as known, we may identify them with $y', y'', y^{(3)}$ and make $y^{(4)}$ equal unity. In the resulting system of partial differential equations of form (1) we shall then have $c = c' = 0$, and system (1) becomes

$$y_{uu} + 2ay_u + 2by_v = 0, \quad y_{vv} + 2a'y_u + 2b'y_v = 0. \quad (4)$$

Let us now consider the familiar Gauss' equations of the metrical theory of surfaces, viz.:

$$\left. \begin{aligned} x_{uu} &= \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} x_v + DX, & x_{uv} &= \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} x_v + D'X, \\ x_{vv} &= \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} x_v + D''X. \end{aligned} \right\} \quad (5)$$

If D and D'' are equal to zero, the curves $u = \text{const.}$ and $v = \text{const.}$ are asymptotic lines, and we observe that the first and third of these equations are now of exactly the same form as equations (4). We can therefore express directly the coefficients of equations (4) in terms of the Christoffel symbols occurring in (5). We shall then be able to change from the homogeneous coordinate system with a semi-covariant tetrahedron of reference to any suitable Cartesian system. The coefficients of these transformations will be functions of the quantities E, F, G and D' and their derivatives with respect to u and v , quantities which are fundamental in the metrical theory of surfaces.

In this paper we shall determine these transformations and study some of the metrical properties of certain geometrical configurations associated with a point on a surface, which have been defined and investigated from a projective point of view by Professor Wilczynski, but which have not as yet

* *Transactions*, Vol. IX, p. 79.

been studied metrically. We can define by means of these configurations some new classes of special points on a surface and some new kinds of surfaces. We shall also apply these notions to a well-known special surface, the Minimal Surface of Enneper.

II. THE FUNDAMENTAL TRANSFORMATIONS OF COORDINATES.

The integrability conditions of system (4) are

$$\begin{aligned} a_v - b'_u &= 0, \\ a'_{uu} - 2a'a_u - 2aa'_u - (a_{vv} + 2b'a_v - 2ba'_v - 4a'b_v) &= 0, \\ b'_{vv} - 2bb'_v - 2b'b'_v - (b'_{uu} + 2ab'_u - 2a'b'_u - 4ba'_u) &= 0. \end{aligned}$$

We shall assume that they are satisfied, so that (4) has four linearly independent solutions $y', y'', y^{(3)}, 1$, which we shall identify with the Cartesian coordinates of a point of a surface.

Comparing equations (4) and the first and third of equations (5) we have the relations

$$a = -\frac{1}{2} \begin{Bmatrix} 1 & 1 \\ 1 & 1 \end{Bmatrix}, \quad a' = -\frac{1}{2} \begin{Bmatrix} 2 & 2 \\ 1 & 1 \end{Bmatrix}, \quad b = -\frac{1}{2} \begin{Bmatrix} 1 & 1 \\ 2 & 2 \end{Bmatrix}, \quad b' = -\frac{1}{2} \begin{Bmatrix} 2 & 2 \\ 2 & 2 \end{Bmatrix}, \quad (6)$$

which express the values of the quantities a, b, a', b' in terms of the Christoffel symbols of the classical theory.

The *homogeneous* Cartesian coordinates of the points P_y, P_z, P_ρ and P_σ are obtained by substituting in order $y', y'', y^{(3)}, y^{(4)}=1$ in each of the expressions (2). The *non-homogeneous* Cartesian coordinates of each of these points are the ratios of the corresponding four homogeneous coordinates concerned. The values obtained by this method for the Cartesian coordinates of the four points $P_y, P_z, P_\rho, P_\sigma$ are given in the following table:

$$\begin{aligned} P_y: & y', y'', y^{(3)}; & P_z: & y' + y'_u/a, y'' + y''_u/a, y^{(3)} + y^{(3)}_u/a; \\ P_\rho: & y' + y'_v/b', y'' + y''_v/b', y^{(3)} + y^{(3)}_v/b'; \\ P_\sigma: & y' + (b'y'_u + ay'_v + y'_{uv})/R, y'' + (b'y''_u + ay''_v + y''_{uv})/R, \\ & y^{(3)} + (b'y^{(3)}_u + ay^{(3)}_v + y^{(3)}_{uv})/R, \end{aligned}$$

where

$$R = \frac{1}{2}(a_v + b'_u + 2ab').$$

In speaking of $y', y'', y^{(3)}$ as the Cartesian coordinates of a point of the surface, we imply that some Cartesian system of coordinates, fixed in space, has been chosen as system of reference. Referred to this fixed Cartesian system, let the coordinates of a general point P of space be $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ and let the coordinates of the same point with reference to the semi-covariant surface

tetrahedron $P_y, P_z, P_\rho, P_\sigma$ be x_1, x_2, x_3, x_4 . Then the relations between the two sets of coordinates of the point P are*

$$\left. \begin{aligned} \omega \bar{\xi} &= x_1 y' + x_2 (y'_u + a y') + x_3 (y'_v + b' y') + x_4 (y'_{uv} + b' y'_u + a y'_v + R y'), \\ \omega \bar{\eta} &= x_1 y'' + x_2 (y''_u + a y'') + x_3 (y''_v + b' y'') + x_4 (y''_{uv} + b' y''_u + a y''_v + R y''), \\ \omega \bar{\zeta} &= x_1 y^{(3)} + x_2 (y^{(3)}_u + a y^{(3)}) + x_3 (y^{(3)}_v + b' y^{(3)}) + x_4 (y^{(3)}_{uv} + b' y^{(3)}_u + a y^{(3)}_v + R y^{(3)}), \\ \omega &= x_1 + x_2 a + x_3 b' + x_4 R, \end{aligned} \right\} (7)$$

where we have written these relations in homogeneous form by introducing a factor of proportionality, ω .

The determinant of the right members of equations (7), which will be designated by Δ , is different from zero for any non-developable surface. As we shall see presently, it is equal to $-HD'$ (see equations (10)).

Let us write $y' = x, y'' = y, y^{(3)} = z$ in agreement with the usual notation for the Cartesian coordinates of a point. Then equations (7) become

$$\left. \begin{aligned} \omega \bar{\xi} &= x_1 x + x_2 (x_u + a x) + x_3 (x_v + b' x) + x_4 (x_{uv} + b' x_u + a x_v + R x), \\ \omega \bar{\eta} &= x_1 y + x_2 (y_u + a y) + x_3 (y_v + b' y) + x_4 (y_{uv} + b' y_u + a y_v + R y), \\ \omega \bar{\zeta} &= x_1 z + x_2 (z_u + a z) + x_3 (z_v + b' z) + x_4 (z_{uv} + b' z_u + a z_v + R z), \\ \omega &= x_1 + x_2 a + x_3 b' + x_4 R. \end{aligned} \right\} (8)$$

If there be substituted in equations (8) the values of x_{uv}, y_{uv}, z_{uv} as given by the second Gauss equation

$$x_{uv} = \begin{Bmatrix} 1 & 2 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 1 & 2 \\ 2 \end{Bmatrix} x_v + D' X,$$

which holds also if y, z and Y, Z be substituted for x and X , respectively, these equations become

$$\left. \begin{aligned} \omega \bar{\xi} &= x_1 x + x_2 (x_u + a x) + x_3 (x_v + b' x) \\ &\quad + x_4 [(b' - 2c)x_u + (a - 2d)x_v + R x + D' X], \\ \omega \bar{\eta} &= x_1 y + x_2 (y_u + a y) + x_3 (y_v + b' y) \\ &\quad + x_4 [(b' - 2c)y_u + (a - 2d)y_v + R y + D' Y], \\ \omega \bar{\zeta} &= x_1 z + x_2 (z_u + a z) + x_3 (z_v + b' z) \\ &\quad + x_4 [(b' - 2c)z_u + (a - 2d)z_v + R z + D' Z], \\ \omega &= x_1 + x_2 a + x_3 b' + x_4 R, \end{aligned} \right\} (9)$$

where

$$c = -\frac{1}{2} \begin{Bmatrix} 1 & 2 \\ 1 \end{Bmatrix}, \quad d = -\frac{1}{2} \begin{Bmatrix} 1 & 2 \\ 2 \end{Bmatrix}, \quad \Delta = -HD', \quad (10)$$

and where X, Y, Z, H are the quantities usually denoted by these letters in the metrical theory of surfaces.

Solving equations (9) for x_1, x_2, x_3, x_4 , we find

$$\left. \begin{aligned} \omega x_1 &= \left\{ \begin{aligned} &SH^2X + D'[(aG - b'F)x_u - (aF - b'E)x_v] \} (\bar{\xi} - x) \\ &+ \{ SH^2Y + D'[(aG - b'F)y_u - (aF - b'E)y_v] \} (\bar{\eta} - y) \\ &+ \{ SH^2Z + D'[(aG - b'F)z_u - (aF - b'E)z_v] \} (\bar{\zeta} - z) - H^2D', \end{aligned} \right. \\ \omega x_2 &= \left\{ \begin{aligned} &(b' - 2c)H^2X - D'(Gx_u - Fx_v) \} (\bar{\xi} - x) \\ &+ \{ (b' - 2c)H^2Y - D'(Gy_u - Fy_v) \} (\bar{\eta} - y) \\ &+ \{ (b' - 2c)H^2Z - D'(Gz_u - Fz_v) \} (\bar{\zeta} - z), \end{aligned} \right. \\ \omega x_3 &= \left\{ \begin{aligned} &(a - 2d)H^2X + D'(Fx_u - Ex_v) \} (\bar{\xi} - x) \\ &+ \{ (a - 2d)H^2Y + D'(Fy_u - Ey_v) \} (\bar{\eta} - y) \\ &+ \{ (a - 2d)H^2Z + D'(Fz_u - Ez_v) \} (\bar{\zeta} - z), \end{aligned} \right. \\ \omega x_4 &= -H^2X(\bar{\xi} - x) - H^2Y(\bar{\eta} - y) - H^2Z(\bar{\zeta} - z), \end{aligned} \right\} \quad (11)$$

where we have put

$$S = \frac{1}{2}(a_v + b'_u - 2ab') + 2(ac + bd).$$

In equations (9) and (11) we have transformations of the kind desired for the purposes of this paper, but we shall find it of advantage to make use also of the moving Cartesian system made up of the surface normal and the lines of curvature tangents of a general point on the surface. We shall speak of this system of coordinate axes as the *surface trihedral*.

The direction cosines of the tangents to the asymptotic curves $v = \text{const.}$ and $u = \text{const.}$ are

$$x_u/\sqrt{E}, y_u/\sqrt{E}, z_u/\sqrt{E} \text{ and } x_v/\sqrt{G}, y_v/\sqrt{G}, z_v/\sqrt{G},$$

respectively. Therefore, the direction cosines of the tangents to the lines of curvature are

$$(x_v\sqrt{E} + x_u\sqrt{G})/2\beta\sqrt{EG}, (y_v\sqrt{E} + y_u\sqrt{G})/2\beta\sqrt{EG}, (z_v\sqrt{E} + z_u\sqrt{G})/2\beta\sqrt{EG}$$

and

$$(x_v\sqrt{E} - x_u\sqrt{G})/2\alpha\sqrt{EG}, (y_v\sqrt{E} - y_u\sqrt{G})/2\alpha\sqrt{EG}, (z_v\sqrt{E} - z_u\sqrt{G})/2\alpha\sqrt{EG},$$

where

$$\alpha = \sqrt{(\sqrt{EG} - F)/2\sqrt{EG}}, \quad \beta = \sqrt{(\sqrt{EG} + F)/2\sqrt{EG}}. \quad (12)$$

Let the coordinates of any point referred to the surface trihedral be ξ, η, ζ . We select the positive directions of these axes in such a way that the trans-

formation of coordinates from the fixed Cartesian system to the surface trihedral is given by the following equations:

$$\left. \begin{aligned} \bar{\xi} &= \frac{x_v \sqrt{E} + x_u \sqrt{G}}{2\beta \sqrt{EG}} \xi + \frac{x_v \sqrt{E} - x_u \sqrt{G}}{2\alpha \sqrt{EG}} \eta + X\zeta + x, \\ \bar{\eta} &= \frac{y_v \sqrt{E} + y_u \sqrt{G}}{2\beta \sqrt{EG}} \xi + \frac{y_v \sqrt{E} - y_u \sqrt{G}}{2\alpha \sqrt{EG}} \eta + Y\zeta + y, \\ \bar{\zeta} &= \frac{z_v \sqrt{E} + z_u \sqrt{G}}{2\beta \sqrt{EG}} \xi + \frac{z_v \sqrt{E} - z_u \sqrt{G}}{2\alpha \sqrt{EG}} \eta + Z\zeta + z. \end{aligned} \right\} \quad (13)$$

Applying equations (13) to (11) we obtain

$$\left. \begin{aligned} \omega x_1 &= \alpha D' (a\sqrt{G} + b'\sqrt{E}) \xi - \beta D' (a\sqrt{G} - b'\sqrt{E}) \eta + HS\zeta - HD', \\ \omega x_2 &= -\alpha D' \sqrt{G} \xi + \beta D' \sqrt{G} \eta + H(b' - 2c)\zeta, \\ \omega x_3 &= -\alpha D' \sqrt{E} \xi - \beta D' \sqrt{E} \eta + H(a - 2d)\zeta, \\ \omega x_4 &= -H\zeta. \end{aligned} \right\} \quad (14)$$

These equations express the relations between the homogeneous coordinates x_1, x_2, x_3, x_4 of a point, referred to the surface tetrahedron P_v, P_z, P_p, P_σ and the Cartesian coordinates ξ, η, ζ of the same point, referred to the surface trihedral composed of the surface normal and the lines of curvature tangents.

III. THE OSCULATING LINEAR COMPLEXES OF THE ASYMPTOTIC CURVES.

The osculating linear complexes of the asymptotic curves were first defined and discussed in detail from the standpoint of projective geometry by Professor Wilczynski.* If we denote by C' and C'' the osculating linear complexes of the asymptotic curves $v = \text{const.}$ and $u = \text{const.}$, respectively, he found the equations of these complexes to be

$$-b_v \bar{\omega}_{34} - b \bar{\omega}_{14} + b \bar{\omega}_{23} = 0 \quad (15)$$

and

$$-a'_u \bar{\omega}_{42} + a' \bar{\omega}_{14} + a' \bar{\omega}_{23} = 0, \quad (16)$$

respectively, where the quantities $\bar{\omega}_{ik}$, the Plückerian homogeneous line coordinates, are connected with the homogeneous point coordinates of the system P_v, P_z, P_p, P_σ in the usual way. We wish to investigate some of the metrical properties of these complexes and shall use the special surface coordinate system which has been defined.

* *Transactions*, Vol. IX, p. 90, et seq.

From equations (14) the transformations for the line coordinates are found to be

$$\left. \begin{aligned} \bar{\omega}_{34} &= -\alpha HD' \sqrt{E} \omega_{31} + \beta HD' \sqrt{E} \omega_{23}, \\ \bar{\omega}_{42} &= \alpha HD' \sqrt{G} \omega_{31} + \beta HD' \sqrt{G} \omega_{23}, \\ \bar{\omega}_{14} &= \alpha HD' (a \sqrt{G} + b \sqrt{E}) \omega_{31} + \beta HD' (a \sqrt{G} - b' \sqrt{E}) \omega_{23} - H^2 D' \omega_{34}, \\ \bar{\omega}_{23} &= HD'^2 \omega_{12} - \alpha HD' [(b' - 2c) \sqrt{E} - (a - 2d) \sqrt{G}] \omega_{31} \\ &\quad + \beta HD' [(b' - 2c) \sqrt{E} + (a - 2d) \sqrt{G}] \omega_{23}, \end{aligned} \right\} \quad (17)$$

where the coordinates ω_{ik} are defined by the relations

$$\left. \begin{aligned} \omega_{14} &= \xi_1 - \xi_2, & \omega_{24} &= \eta_1 - \eta_2, & \omega_{34} &= \zeta_1 - \zeta_2, \\ \omega_{23} &= \eta_1 \zeta_2 - \eta_2 \zeta_1, & \omega_{31} &= \xi_2 \zeta_1 - \xi_1 \zeta_2, & \omega_{12} &= \xi_1 \eta_2 - \xi_2 \eta_1; \end{aligned} \right\} \quad (18)$$

ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 being two points of the line.

By transformations (17) the equation of complex C' , equation (15), becomes

$$bD' \omega_{12} - \alpha M_1 \omega_{31} + \beta M_2 \omega_{23} + bH \omega_{34} = 0, \quad (19)$$

where

$$M_1 = M' \sqrt{E} + 2bd \sqrt{G}, \quad M_2 = M' \sqrt{E} - 2bd \sqrt{G}, \quad M' = 2bb' - 2bc - b_v. \quad (20)$$

If we use the quantities ω_{ik} as given in (18), the equation of any linear complex may be written in the form

$$a_{12} \omega_{12} + a_{31} \omega_{31} + a_{14} \omega_{14} + a_{23} \omega_{23} + a_{34} \omega_{34} + a_{24} \omega_{24} = 0,$$

where we have used the homogeneous notation for the coefficients also. The equations of the axis of the complex will then be*

$$\frac{\xi - \frac{a_{24} a_{12} - a_{34} a_{31}}{a_{23}^2 + a_{31}^2 + a_{12}^2}}{a_{23}} = \frac{\eta - \frac{a_{34} a_{23} - a_{14} a_{12}}{a_{23}^2 + a_{31}^2 + a_{12}^2}}{a_{31}} = \frac{\zeta - \frac{a_{14} a_{31} - a_{24} a_{23}}{a_{23}^2 + a_{31}^2 + a_{12}^2}}{a_{12}}. \quad (21)$$

Therefore, from (21) the equations of the axis of the complex C' are

$$\frac{\xi - (\alpha b H M_1 / V'^2)}{\beta M_2} = \frac{\eta - (\beta b H M_2 / V'^2)}{-\alpha M_1} = \frac{\zeta}{b D'}, \quad (22)$$

where

$$V'^2 = \alpha^2 M_1^2 + \beta^2 M_2^2 + b^2 D'^2 = M'^2 E - 4bd FM' + 4b^2 d^2 G + b^2 D'^2. \quad (23)$$

We shall call the plane through the surface point perpendicular to the axis of the complex the *principal plane*, and its intersection with the tangent plane the *principal line*. The equation of the principal plane with respect to the complex C' is found to be

$$\beta M_2 \xi - \alpha M_1 \eta + b D' \zeta = 0, \quad (24)$$

* Plücker, "Neue Geometrie des Raumes," p. 32.

and, therefore, the equations of the principal line are

$$\beta M_2 \xi - \alpha M_1 \eta = 0, \quad \zeta = 0. \quad (25)$$

The point on the principal line where the axis of the complex intersects the tangent plane is at a distance δ' from the surface point, where

$$\delta' = bHU'/V'^2 \quad \text{and} \quad U'^2 = \alpha^2 M_1^2 + \beta^2 M_2^2. \quad (26)$$

In order to find the *parameter* of the complex we change from the surface trihedral to another rectangular system, ξ', η', ζ' , whose origin is the point of intersection of the axis of the complex with the tangent plane, the ξ' axis coinciding with the principal line and the ζ' axis with the axis of the complex.

From equations (25) and (22) the direction cosines of the ξ' and ζ' axes are

$$\alpha M_1/U', \beta M_2/U', 0, \quad \text{and} \quad \beta M_2/V', -\alpha M_1/V', bD'/V',$$

respectively. Consequently, the direction cosines of the η' axis are

$$-\beta bD'M_2/U'V', \alpha bD'M_1/U'V', U'/V',$$

the matter of the positive direction of the lines being left arbitrary. Therefore, the equations of transformation from the surface trihedral to the special coordinate system just defined are

$$\begin{aligned} \xi &= (\alpha M_1/U')\xi' - (\beta bD'M_2/U'V')\eta' + (\beta M_2/V')\zeta' + (abHM_1/V'^2), \\ \eta &= (\beta M_2/U')\xi' + (\alpha bD'M_1/U'V')\eta' - (\alpha M_1/V')\zeta' + (\beta bHM_2/V'^2), \\ \zeta &= (U'/V')\eta' + (bD'/V')\zeta'. \end{aligned}$$

The corresponding transformations for the line coordinates are found to be

$$\left. \begin{aligned} \omega_{12} &= (bD'/V')\omega'_{12} + (U'/V')\omega'_{31} - (b^2HD'U'/V'^3)\omega'_{24} + (bHU'^2/V'^3)\omega'_{34}, \\ \omega_{23} &= (\beta M_2/V')\omega'_{12} - (\beta bD'M_2/U'V')\omega'_{31} + (\alpha M_1/U')\omega'_{23} \\ &\quad - (\beta bHM_2U'/V'^3)\omega'_{24} - (\beta b^2HD'M_2/V'^3)\omega'_{34}, \\ \omega_{34} &= (U'/V')\omega'_{24} + (bD'/V')\omega'_{34}, \\ \omega_{31} &= -(\alpha M_1/V')\omega'_{12} + (\alpha bD'M_1/U'V')\omega'_{31} + (\beta M_2/U')\omega'_{23} \\ &\quad + (abHM_1U'/V'^3)\omega'_{24} + (\alpha b^2HD'M_1/V'^3)\omega'_{34}. \end{aligned} \right\} \quad (27)$$

Substituting the values (27) in equation (19) we have as the transformed equation of the complex the equation

$$\omega'_{12} + (b^2HD'/V'^2)\omega'_{34} = 0.$$

Therefore, the parameter of complex C' is

$$P' = b^2HD'/V'^2. \quad (28)$$

We now consider the corresponding properties of the complex C'' . The equation of this complex referred to the surface trihedral is

$$a' D' \omega_{12} + \alpha N_1 \omega_{31} + \beta N_2 \omega_{23} - a' H \omega_{34} = 0, \quad (29)$$

where

$$N_1 = N' \sqrt{G} + 2a' c \sqrt{E}, \quad N_2 = N' \sqrt{G} - 2a' c \sqrt{E}, \quad N' = 2aa' - 2a'd - a''_u, \quad (30)$$

and the equations of its axis are

$$\frac{\xi - (\alpha a' H N_1 / V''^2)}{\beta N_2} = \frac{\eta + (\beta a' H N_2 / V''^2)}{\alpha N_1} = \frac{\zeta}{a' D'}, \quad (31)$$

where

$$V''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2 + a'^2 D'^2 = N'^2 G - 4a' c F N' + 4a'^2 c^2 E + a'^2 D'^2. \quad (32)$$

The principal plane with respect to the complex C'' is given by

$$\beta N_2 \xi + \alpha N_1 \eta + a' D' \zeta = 0,$$

and, therefore, the principal line with respect to C'' has the equations

$$\beta N_2 \xi + \alpha N_1 \eta = 0, \quad \zeta = 0.$$

The distance from the surface point to the point in which the axis of this complex intersects the tangent plane is given by

$$\delta'' = a' H U'' / V''^2, \quad \text{where } U''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2. \quad (33)$$

We make a similar change of coordinate system in order to find the parameter of complex C'' as in the case of complex C' . The transformed equation of the complex is

$$\omega''_{12} - (a'^2 H D' / V''^2) \omega''_{34} = 0,$$

so that the parameter has the value

$$P'' = -a'^2 H D' / V''^2. \quad (34)$$

Comparing the parameters P' and P'' of the two complexes we see that their values are opposite in sign, if the surface and the two asymptotic curves concerned are real. Therefore, *the linear complexes which osculate two real asymptotic curves of a real surface at their point of intersection are oppositely twisted.*

If ϕ is the angle between the axes of the complexes C' and C'' and if K denotes the perpendicular distance between them, we find from equations (22) and (31) that these quantities are given by

$$\cos \phi = Q^2 / V' V'', \quad (35)$$

$$K = H D' (a'^2 V'^2 - b^2 V''^2) Q^2 / V'^2 V''^2 \sqrt{V'^2 V''^2 - Q^4}, \quad (36)$$

where

$$Q^2 = \beta^2 M_2 N_2 - \alpha^2 M_1 N_1 + b a' D'^2. \quad (37)$$

IV. THE PENCIL OF LINEAR COMPLEXES DETERMINED BY THE COMPLEXES C' AND C'' .

The equation of any complex C_λ of the pencil determined by C' and C'' is a linear combination of equations (19) and (29) and is, therefore, of the form

$$D'(b + \lambda a')\omega_{12} - \alpha(M_1 - \lambda N_1)\omega_{31} + \beta(M_2 + \lambda N_2)\omega_{23} + H(b - \lambda a')\omega_{34} = 0. \quad (38)$$

The equation of the principal plane with respect to any complex C_λ of the pencil is

$$\beta(M_2 + \lambda N_2)\xi - \alpha(M_1 - \lambda N_1)\eta + D'(b + \lambda a')\zeta = 0,$$

and, therefore, the equations of the principal line with respect to C_λ are

$$\beta(M_2 + \lambda N_2)\xi - \alpha(M_1 - \lambda N_1)\eta = 0, \quad \zeta = 0. \quad (39)$$

The line conjugate to the surface normal with respect to C_λ is given by the equations

$$\alpha(M_1 - \lambda N_1)\xi + \beta(M_2 + \lambda N_2)\eta = 0, \quad \zeta = 0. \quad (40)$$

All of these lines form a pencil in the tangent plane whose vertex is the point

$$\xi = \beta P_2/A, \quad \eta = -\alpha Q_2/A, \quad \zeta = 0,$$

where we have put

$$A = M'N' - 4ba'cd, \quad (41)$$

and

$$\left. \begin{aligned} P_2 &= a'\sqrt{E}(M' - 2bc) + b\sqrt{G}(N' - 2a'd), \\ Q_2 &= a'\sqrt{E}(M' - 2bc) - b\sqrt{G}(N' - 2a'd). \end{aligned} \right\} \quad (42)$$

Corresponding lines of the pencils given by equations (39) and (40) are perpendicular to each other. Therefore, the locus of their points of intersection is a circle. Its equations are

$$\xi^2 + \eta^2 - (\beta P_2/A)\xi + (\alpha Q_2/A)\eta = 0, \quad \zeta = 0.$$

We may state the theorem:

The line which corresponds to the surface normal in any one of the complexes C_λ of the pencil is in the tangent plane of the corresponding surface point and perpendicular to the principal line with respect to that complex.

The special complexes of the pencil (38) are given by the values of λ which satisfy the equation of the form

$$a_{12}a_{34} + a_{31}a_{42} + a_{14}a_{23} = 0.$$

These values are $-b/a'$ and b/a' . We shall call the corresponding special complexes the special complexes of the first and second kind, respectively.

If we put $\lambda = -b/a'$ in equation (38) we obtain the equation of the special complex of the first kind. It is

$$\alpha P_1 \omega_{31} - \beta Q_1 \omega_{23} - 2ba' H \omega_{34} = 0, \quad (43)$$

where

$$\left. \begin{aligned} P_1 &= a' \sqrt{E} (M' + 2bc) + b \sqrt{G} (N' + 2a'd), \\ Q_1 &= a' \sqrt{E} (M' + 2bc) - b \sqrt{G} (N' + 2a'd). \end{aligned} \right\} \quad (44)$$

The equations of the axis of this complex, or of the directrix of the first kind, are found to be

$$\frac{\xi - (2\alpha b a' H P_1 / V_1^2)}{\beta Q_1} = \frac{\eta - (2\beta b a' H Q_1 / V_1^2)}{-\alpha P_1} = \frac{\zeta}{0}, \quad (45)$$

where

$$V_1^2 = \alpha^2 P_1^2 + \beta^2 Q_1^2 = a'^2 V'^2 + b^2 V''^2 - 2ba' Q^2. \quad (46)$$

The equation of the special complex of the second kind is

$$2ba' D' \omega_{12} - \alpha Q_2 \omega_{31} + \beta P_2 \omega_{23} = 0, \quad (47)$$

and the equations of its axis, or of the directrix of the second kind, are

$$\xi / \beta P_2 = \eta / -\alpha Q_2 = \zeta / 2ba' D', \quad (48)$$

in connection with which equations we introduce the quantity V_2 defined by

$$V_2^2 = \alpha^2 Q_2^2 + \beta^2 P_2^2 + 4b^2 a'^2 D'^2 = a'^2 V'^2 + b^2 V''^2 + 2ba' Q^2. \quad (49)$$

Equations (45) and (48) enable us to compute a number of important quantities. Let ψ be the angle between the two directrices, I the perpendicular distance between them, l, m, n the direction cosines of their common perpendicular, ξ, η, ζ the coordinates of the middle point of the congruence. Then we shall have

$$\left. \begin{aligned} \cos \psi &= (a'^2 V'^2 - b^2 V''^2) / V_1 V_2, \quad I = 2ba' H D' / V_4, \\ l &= \alpha D' P_1 / V_4, \quad m = \beta D' Q_1 / V_4, \quad n = -H A / V_4, \\ \xi &= H (\alpha b a' D'^2 P_1 + \beta H A P_2) / V_4^2, \quad \eta = H (\beta b a' D'^2 Q_1 - \alpha H A Q_2) / V_4^2, \\ \zeta &= b a' H^2 D' A / V_4^2, \end{aligned} \right\} \quad (50)$$

where

$$V_4^2 = V_1^2 D'^2 + H^2 A^2.$$

V. THE CYLINDROID FORMED BY THE AXES OF THE LINEAR COMPLEXES OF THE PENCIL DETERMINED BY THE COMPLEXES C' AND C'' .

The ruled surface formed by the axes of a pencil of linear complexes is a special ruled surface of the third degree, usually known as a cylindroid. In order to obtain a simple form for the equation of the cylindroid, which belongs to the pencil of linear complexes C_λ , we introduce a new rectangular system of coordinates, whose origin is the middle point of the congruence,

whose ζ''' axis is the common perpendicular of the two directrices, and whose ξ''' and η''' axes bisect the angles made by the lines parallel to the directrices through the origin.

From equations (45) and (48) we find the direction cosines of the ξ''' and η''' axes to be

$\beta S_1/W_1$, $-\alpha R_1/W_1$, $2ba'D'V_1/W_1$ and $\beta S_2/W_2$, $-\alpha R_2/W_2$, $-2ba'D'V_1/W_2$, respectively, where

$$\begin{aligned} V_3^2 &= \alpha^2 P_1 Q_2 + \beta^2 P_2 Q_1, \\ S_1 &= Q_1 V_2 + P_2 V_1, \quad S_2 = Q_1 V_2 - P_2 V_1, \\ R_1 &= P_1 V_2 + Q_2 V_1, \quad R_2 = P_1 V_2 - Q_2 V_1, \\ W_1^2 &= 2V_1 V_2 (V_1 V_2 + V_3^2), \quad W_2^2 = 2V_1 V_2 (V_1 V_2 - V_3^2). \end{aligned}$$

The direction cosines of the ζ''' axis were given among the relations (50). The positive directions in regard to these new coordinate axes are left arbitrary. Then the necessary transformations of the point coordinates from the surface trihedral to this new system of coordinates are given by

$$\begin{aligned} \xi &= (\beta S_1/W_1)\xi''' + (\beta S_2/W_2)\eta''' + (\alpha D'P_1/V_4)\zeta''' + H(\alpha ba'D'^2 P_1 + \beta H A P_2)/V_4^2, \\ \eta &= (-\alpha R_1/W_1)\xi''' - (\alpha R_2/W_2)\eta''' + (\beta D'Q_1/V_4)\zeta''' + H(\alpha ba'D'^2 Q_1 - \beta H A Q_2)/V_4^2, \\ \zeta &= (2ba'D'V_1/W_1)\xi''' - (2ba'D'V_1/W_2)\eta''' - (H A/V_4)\zeta''' + ba'H^2 D'A/V_4^2. \end{aligned}$$

We shall not write down the corresponding transformations of the line coordinates.

We find the equations of the complexes C' and C'' , referred to this new system of coordinates, to be

$$\begin{aligned} [ba'D'H(V_2+V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2-V_1)/W_2]\omega_{24}''' \\ + [(V_2+V_1)V_4/W_2]\omega_{23}''' - [(V_2-V_1)V_4/W_1]\omega_{31}''' = 0, \end{aligned} \quad (51)$$

and

$$\begin{aligned} [ba'D'H(V_2-V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2+V_1)/W_2]\omega_{24}''' \\ + [(V_2-V_1)V_4/W_2]\omega_{23}''' - [(V_2+V_1)V_4/W_1]\omega_{31}''' = 0, \end{aligned} \quad (52)$$

respectively.

According to the general theory of a pencil of linear complexes, when the equations of the fundamental complexes are in the form of equations (51) and (52), the equation of the cylindroid formed by the axes of the complexes of the pencil is of the form*

$$(\xi'''^2 + \eta'''^2)\zeta''' = (a_{14}/a_{23} - a_{24}/a_{31})\xi'''\eta''',$$

a_{14} , a_{23} , a_{24} , a_{31} being taken from the equation of either of the fundamental complexes. We find from either equation (51) or (52) that

$$\frac{a_{14}}{a_{23}} = ba'D'HW_2/V_4W_1 \quad \text{and} \quad \frac{a_{24}}{a_{31}} = -ba'D'HW_1/V_4W_2.$$

* Plücker, "Neue Geometrie des Raumes," p. 97.

Therefore, the equation of the cylindroid in this case is

$$(\xi'''^2 + \eta'''^2)\zeta''' = (4ba'D'HV_1^2V_2^2/W_1W_2V_4)\xi'''\eta'''. \quad (53)$$

If we let

$$T = 2ba'D'HV_1^2V_2^2/W_1W_2V_4,$$

then the cylindroid lies between the two planes

$$\zeta''' = T \text{ and } \zeta''' = -T;$$

that is, these planes are the tangent planes at the pinch points,

$$\xi''' = 0, \eta''' = 0, \zeta''' = T; \quad \xi''' = 0, \eta''' = 0, \zeta''' = -T.$$

VI. THE DIRECTRIX CURVES.

Professor Wilczynski has shown* that there exist, in general, two one-parameter families of curves on the surface S such that, if the point P describes one of these curves, both of its directrices will simultaneously describe developable surfaces. He called these curves *directrix curves*. The quadratic equation which determines the tangents of the two directrix curves which pass through the point P he found to be†

$$bL\delta u^2 + 2M\delta u\delta v - a'N\delta v^2 = 0,$$

where

$$\left. \begin{aligned} L &= -2a'(2ba'f + 2abb_v + ba'_{uv}) + ba_u'^2, \\ M &= 2ba'(a'b_{uv} - ba'_{uv}) + 2b^2a'_ua'_v - 2a'^2b_ub_v, \\ N &= -2b(2ba'g + 2ba'a'_u + a'b_{vv}) + a'b_v'^2, \end{aligned} \right\} \quad (54)$$

the surface S having been defined by equations (3). We wish to find the corresponding equation when the surface is defined by equations (1).

Equations (1) may be changed into equations (3) by a transformation of the form $y = \lambda\bar{y}$. Then the quantities f and g in equations (3) have the values‡

$$f = c - a_u - a^2 - 2bb', \quad g = c' - b'_v - b'^2 - 2aa'.$$

If in these expressions we make $c = c' = 0$ and substitute the resulting values of f and g in the quantities (54), the corresponding equation of the directrix curves can be put in the form

$$b^2(C^2 + 2a'C_u + 4a'^2B)\delta u^2 - 2[a'^2(bB_u - Bb_u) - b^2(a'C_v - Ca'_v)]\delta u\delta v - a'^2(B^2 + 2bB_v + 4b^2C)\delta v^2 = 0, \quad (55)$$

in which the quantities B and C have the values

$$B = M' + 2bc, \quad C = N' + 2a'd.$$

* Transactions, Vol. IX, p. 114, et seq.

† Transactions, Vol. IX, p. 116.

‡ Transactions, Vol. VIII, p. 246.

VII. SPECIAL POINTS.

Since the pencil of linear complexes formed from the osculating linear complexes of the asymptotic lines plays an important part in the study made in this paper, and since these complexes are indeterminate if a' or b is equal to zero, it is clear that we have tacitly assumed that neither a' nor b shall vanish. Therefore, if the surface contains straight lines, all points upon such straight lines are to be excluded and, of course, all ruled surfaces are also excluded from consideration.

We wish to discuss the values of K , $\cos \phi$ and $\cos \psi$ in (36), (35) and (50), respectively. These quantities are seen to be indeterminate in value when either a' or b is zero. If there are other values of the parameters different from those that make either a' or b zero, for which these quantities become indeterminate, the corresponding points must also be excluded. Thus, the developments of this chapter apply only to such points for which the quantity Q^2 or $a'^2 V'^2 - b^2 V''^2$ is equal to zero while neither V' , nor V'' nor $V'^2 V''^2 - Q^4$ is equal to zero.

a) *Points for Which the Axes of the Complexes C' and C'' Intersect.*

The axes of these complexes intersect when K is zero. This quantity is zero under the two separate conditions,

$$Q^2 = 0 \tag{56}$$

and

$$a'^2 V'^2 - b^2 V''^2 = 0. \tag{57}$$

We observe that equation (56) makes the quantity $\cos \phi$ also zero and that $\cos \psi$ is zero under the condition (57). Therefore, we can state the following theorems:

If the axes of the osculating linear complexes of the asymptotic lines are determinate and intersect, either these axes are perpendicular to each other or the directrices of the first and second kind are perpendicular to each other, or both conditions may be fulfilled at the same time.

If the axes of the osculating linear complexes of the asymptotic lines are determinate and perpendicular to each other, they intersect.

If the directrices of the first and second kind are perpendicular to each other, the axes of the osculating linear complexes of the asymptotic lines intersect if they are determinate.

b) *Points for Which the Directrices of the First and Second Kind Are Perpendicular to Each Other.*

Such points were defined by equation (57). They are of particular interest on account of the further specialization which exists in the geometrical configuration to which the directrices belong and whose metrical properties we have developed.

If we consider the condition that the parameters P' and P'' of the complexes C' and C'' be numerically equal, we find from (28) and (34) that it is exactly equation (57). We have, therefore, the theorem:

If the asymptotic tangents of a point of a surface are real and if its directrices are perpendicular to each other, the parameters of the osculating linear complexes of the corresponding asymptotic lines are numerically equal and opposite in sign, and conversely.

The equation of the cylindroid reduces to

$$(\xi'''^2 + \eta'''^2)\zeta''' = (2ba'D'/V_4)\xi'''\eta'''.$$

We see that the directrices are in this case the singular tangents to the cylindroid. Therefore, *when the directrices of the first and second kind are perpendicular to each other, they are the singular tangents to the cylindroid.*

Since the directrix of the first kind is in the tangent plane and in this case the two directrices are perpendicular to each other, the plane of the directrix of the second kind and the axis of the cylindroid must be perpendicular to the tangent plane. It is found to bisect the angle the axes of the complexes C' and C'' make with each other. Consequently, these axes must meet the tangent plane in points which are at the same distance from the surface point; that is, δ' and δ'' in (26) and (33), respectively, must be equal, which fact may easily be verified. We may state the theorem:

When the axes of the complexes C' and C'' intersect each other and the directrices of the first and second kind are perpendicular to each other, the axes make the same angle with the tangent plane and meet it in points equidistant from the surface point.

Let us write equation (57) in the form

$$(a'^2M'^2E - b^2N'^2G) - 4ba'F(a'dM' - bcN') + 4a'^2b^2(d^2G - c^2E) = 0.$$

Evidently, it may be satisfied by means of the three separate equations

$$\left. \begin{aligned} (a'M'\sqrt{E} - bN'\sqrt{G})(a'M'\sqrt{E} + bN'\sqrt{G}) &= 0, & a'dM' - bcN' &= 0, \\ (d\sqrt{G} - c\sqrt{E})(d\sqrt{G} + c\sqrt{E}) &= 0. \end{aligned} \right\} \quad (58)$$

These may be divided into two sets, each of three separate equations, any two of which imply the third. They are

$$a'M'\sqrt{E}-bN'\sqrt{G}=0, \quad d\sqrt{G}-c\sqrt{E}=0, \quad a'dM'-bcN'=0, \quad (59)$$

and

$$a'M'\sqrt{E}+bN'\sqrt{G}=0, \quad d\sqrt{G}+c\sqrt{E}=0, \quad a'dM'-bcN'=0. \quad (60)$$

We observe that in the case given by equations (59) the directrix of the first kind is perpendicular to one of the principal normal planes, that consequently the directrix of the second kind and the axis of the cylindroid are in this plane, and that in the case of equations (60) a similar position exists with reference to the other principal normal plane.

c) *Points for Which the Directrix of the Second Kind Coincides with the Normal to the Surface.*

According to equation (48) the directrix of the second kind coincides with the normal to the surface if and only if P_2 and Q_2 separately are equal to zero; that is, if and only if

$$M' = 2bc, \quad N' = 2a'd. \quad (61)$$

Since for these points the directrices must be perpendicular to each other, these points must be included among the special points which were defined by equation (57). The fact that equation (57) is satisfied by equations (61) may easily be verified.

If in addition to equations (61) the equation

$$d\sqrt{G}-c\sqrt{E}=0$$

is satisfied, the directrix of the first kind is perpendicular to one of the principal normal planes, while if the additional equation which is satisfied is

$$d\sqrt{G}+c\sqrt{E}=0,$$

the same fact holds in regard to the other principal normal plane.

If the relations (61) be substituted in the quantities B and C which occur in equation (55), that equation of the directrix curves reduces to the form

$$E\delta u^2 - G\delta v^2 = 0,$$

so that at these points the directrix curves coincide with the lines of curvature, as they obviously should.

By introducing in equations (61) the values of M' and N' from (20) and (30), respectively, we may write them in the form

$$b_v - 2bb' + 4bc = 0, \quad a_u - 2aa' + 4a'd = 0. \quad (62)$$

If in the two general relations

$$\frac{H_u}{H} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}, \quad \frac{H_v}{H} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix},$$

we replace the Christoffel symbols by their values in a, b, c, d we obtain

$$c = -b' - H_v/2H, \quad d = -a - H_u/2H. \quad (63)$$

By means of these relations then, equations (62) become

$$Hb_v - 2bH_v - 6Hbb' = 0, \quad Ha'_u - 2a'H_u - 6Haa' = 0.$$

The conditions developed in this chapter characterize certain special kinds of special points on a surface if they are merely satisfied by special values of the parameters. They characterize special classes of surfaces if they can be satisfied identically. The existence, in general, of real points or surfaces of these several kinds will not be discussed in this paper. In the next chapter we shall show however by means of a particular surface that points of the several different kinds which have been defined actually exist.

VIII. THE MINIMAL SURFACE OF ENNEPER.

In order that the significance of the preceding developments may appear more clearly we proceed to apply them to a well-known surface, the Minimal Surface of Enneper. We shall obtain, in this way, some new properties of this surface.

The Minimal Surface of Enneper is represented by the equations

$$x = \frac{1}{4}(u+v)(u^2 - 4uv + v^2 + 6), \quad y = \frac{1}{4}(u-v)(u^2 + 4uv + v^2 + 6), \quad z = 3uv, \quad (64)$$

where the curves $u = \text{const.}$ and $v = \text{const.}$ are the asymptotic lines, the lines of curvature being given by the equations $u+v = \text{const.}$ and $u-v = \text{const.}$

The fundamental quantities E, F, G, H, D' and the quantities α, β , defined in (12), are found to have the values,

$$E = \frac{9}{8}W^2, \quad F = 0, \quad G = \frac{9}{8}W^2, \quad H = \frac{9}{8}W^2, \quad D' = -3, \quad \alpha = \frac{1}{\sqrt{2}}, \quad \beta = \frac{1}{\sqrt{2}},$$

where

$$W = u^2 + v^2 + 2.$$

The quantities a, a', b, b', c, d which occur in (6) and (10) are,

$$a = -u/W, \quad a' = u/W, \quad b = v/W, \quad b' = -v/W, \quad c = -v/W, \quad d = -u/W,$$

We see from the values just given that the condition that neither a' nor b shall be zero implies for this surface that neither u nor v shall be equal to zero. The zero values of the parameters give the intersection of the surface with the XY -plane, which is made up of the two straight lines, $x+y=0$, $x-y=0$.

Therefore, every point upon these two straight lines is excluded from our considerations.

By easy substitutions we obtain the following equations and results in connection with the complexes C' and C'' , the surface coordinate system being used:

Equation of complex C' ,

$$8v\omega_{12} - 2(W - 2v^2 + 2uv)\omega_{31} + 2(W - 2v^2 - 2uv)\omega_{23} - 3vW^2\omega_{34} = 0;$$

Equation of complex C'' ,

$$8u\omega_{12} + 2(W - 2u^2 + 2uv)\omega_{31} + 2(W - 2u^2 - 2uv)\omega_{23} + 3uW^2\omega_{34} = 0;$$

Equations of axis of complex C' ,

$$\frac{\xi + \frac{3}{4}v[W + 2v(u-v)]}{-[W - 2v(u+v)]} = \frac{\eta + \frac{3}{4}v[W - 2v(u+v)]}{[W + 2v(u-v)]} = \frac{\zeta}{4v}; \quad (65)$$

Equations of axis of complex C'' ,

$$\frac{\xi + \frac{3}{4}u[W - 2u(u-v)]}{[W - 2u(u+v)]} = \frac{\eta - \frac{3}{4}u[W - 2u(u+v)]}{[W - 2u(u-v)]} = \frac{\zeta}{4u}; \quad (66)$$

The parameters P' and P'' are given by

$$P' = -3v^2, \quad P'' = 3u^2.$$

We find that the expression Q^2 in (56) is identically zero except when the parameters have the value zero and that each of the two quantities V' and V'' has the value $9/8$. No points then are excluded by the limitation that was placed upon the quantities V' , V'' and Q^2 . Therefore, every general point on the surface is of the type defined by equation (56), that is, for every such point the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other.

The equations of the special complexes of the first and second kind and of the two directrices are as follows:

Equation of special complex of first kind,

$$(u+v)\omega_{31} - (u-v)\omega_{23} + 3uvW\omega_{34} = 0;$$

Equation of special complex of second kind,

$$8uv\omega_{12} - (u-v)(W + 4uv)\omega_{31} + (u+v)(W - 4uv)\omega_{23} = 0;$$

Equations of directrix of first kind,

$$\frac{\xi + (3uv(u+v)W/2(u^2+v^2))}{u-v} = \frac{\eta + (3uv(u-v)W/2(u^2+v^2))}{-(u+v)} = \frac{\zeta}{0}; \quad (67)$$

Equations of directrix of second kind,

$$\frac{\xi}{(u+v)(W-4uv)} = -\frac{\eta}{(u-v)(W+4uv)} = \frac{\zeta}{8uv}. \quad (68)$$

The expressions for the several quantities in (50) and (54) are given by,

$$\left. \begin{aligned} \cos \psi &= (u-v)(u+v)/(u^2+v^2), \quad I=6uv, \quad l=2(u+v)/W, \\ m &= 2(u-v)/W, \quad \eta = (W-4)/W, \\ \xi &= 3(u+v)[(u-v)^4 - 4(u^2v^2+1)]/4W, \\ \eta &= -3(u-v)[(u+v)^4 - 4(u^2v^2+1)]/4W, \\ \zeta &= 3uv(W-4)/W, \quad T = -3(u^2+v^2)/2. \end{aligned} \right\} \quad (69)$$

Equation (57) here assumes the form

$$(u-v)(u+v)=0.$$

Therefore, the directrices are perpendicular to each other at the points given by the two conditions, $u=v \neq 0$ and $u=-v \neq 0$. But the condition $u=v \neq 0$ satisfies equations (59) for this surface and the condition $u=-v \neq 0$ equations (60). Therefore, the points given by these two conditions are of the more special types defined by equations (59) and (60).

The curves defined on the surface by the two conditions $u=v \neq 0$ and $u=-v \neq 0$ are found from (64) to be given by

$$x^2 = \frac{z}{27} (9-z)^2, \quad y=0, \quad (70)$$

and

$$x=0, \quad y^2 = \frac{-z}{27} (9+z)^2, \quad (71)$$

respectively. Every point of these two plane cubic curves, except the origin, is of the special kind described.

For this surface the two equations (61) reduce to

$$u^2 - 3v^2 + 2 = 0, \quad 3u^2 - v^2 - 2 = 0,$$

which have the solutions $u = \pm 1$ and $v = \pm 1$. These values of the parameters give four points of this very special type, for which the directrix of the second kind coincides with the normal to the surface. The points are

$$\begin{aligned} x &= 2, & x &= -2, & x &= 0, & x &= 0, \\ y &= 0, & y &= 0, & y &= 2, & y &= -2, \\ z &= 3, & z &= 3, & z &= -3, & z &= -3. \end{aligned}$$

Of course these points are on the curves given in (70) and (71) and are found to be the maximum and minimum points on the ovals of those curves with reference to their axes of symmetry.

Equation (55), the differential equation of the directrix curves, becomes $\delta u^2/u^2 - \delta v^2/v^2 = 0$, so that these curves are given by $uv = c_1$ and $u/v = c_2$. They may be represented parametrically by

$$\begin{aligned}x &= (u^2 + c_1) [u^4 + (6 - 4c_1)u^2 + c_1^2] / 4u^3, \\y &= (u^2 - c_1) [u^4 + (6 + 4c_1)u^2 + c_1^2] / 4u^3, \\z &= 3c_1,\end{aligned}$$

and

$$\begin{aligned}x &= u(1 + c_2) [u^2(1 - 4c_2 + c_2^2) + 6c_2^2] / 4c_2^3, \\y &= u(1 - c_2) [u^2(1 + 4c_2 + c_2^2) + 6c_2^2] / 4c_2^3, \\z &= 3u^2/c_2,\end{aligned}$$

respectively.

We observe that the directrix curve directions are conjugate, that along a curve given by $uv = c_1$, a plane curve, the two directrices are a constant distance apart and that for the points on the two curves (70) and (71) the directrix curve directions are the same as those of the lines of curvature.

It has been found that in general the axes of the two complexes C' and C'' , whose equations were given in (72) and (73) intersect. Their point of intersection proves to be the middle point of the congruence, the point ξ, η, ζ given in (69), which we will call the point C .

The two directrices whose equations were given in (67) and (68) intersect the axis of the cylindroid, the line which goes through the point ξ, η, ζ and has the direction cosines l, m, n , as given in (69), in the points C_1 and C_2 which are as follows:

Point C_1 ,

$$\xi = \frac{3}{4}(u+v)[W-4(uv+1)], \quad \eta = -\frac{3}{4}(u-v)[W+4(uv-1)], \quad \zeta = 0,$$

Point C_2 ,

$$\begin{aligned}\xi &= 3(u+v)(W-4)(W-4uv)/4W, \\ \eta &= -3(u-v)(W-4)(W+4uv)/4W, \\ \zeta &= 6uv(W-4)/W.\end{aligned}$$

As the point P traces the surface each of the three points C, C_1, C_2 traces a surface which is covariant with the original surface.

From equations (13) the equations of transformation from the surface trihedral to the fixed trihedral, with reference to which the equations of the surface are given, are found to be

$$\begin{aligned}\bar{\xi} &= \frac{-2(uv-1)}{W} \xi - \frac{(u^2-v^2)}{W} \eta + \frac{2(u+v)}{W} \zeta + x, \\ \bar{\eta} &= \frac{(u^2-v^2)}{W} \xi - \frac{2(uv+1)}{W} \eta - \frac{2(u-v)}{W} \zeta + y, \\ \bar{\zeta} &= \frac{2(u+v)}{W} \xi + \frac{2(u-v)}{W} \eta + \frac{(W-4)}{W} \zeta + z.\end{aligned}$$

By means of these equations we find that the coordinates of the points C, C_1, C_2 , referred to the fixed coordinate system, have the following form:

$$\begin{array}{lll} C. & x=u^3+v^3, & C_1. & x=u^3+v^3, & C_2. & x=u^3+v^3, \\ & y=u^3-v^3, & & y=u^3-v^3, & & y=u^3-v^3, \\ & z=0, & & z=-3uv, & & z=3uv. \end{array}$$

These expressions for the coordinates of these points show several important relations: That the axis of the cylindroid is always perpendicular to the XY -plane, and consequently the axes of the complexes and the directrices are always parallel to this plane; that the middle point of the congruence is in the XY -plane and, therefore, the axes of the complexes are in this plane; that the covariant surfaces of the points C_1 and C_2 are exactly like in character, being reciprocally symmetrical surfaces with references to the XY -plane.

The covariant surfaces of the points C_1 and C_2 are the simple cubic surfaces

$$x^2-y^2=-\frac{4}{27}z^3, \text{ and } x^2-y^2=\frac{4}{27}z^3,$$

respectively. These surfaces have in common the lines $x+y=0$ and $x-y=0$ in the XY -plane, lines which are upon the Surface of Enneper itself and the points of which were excluded from our considerations.

It is easy to show that if the point P moves along a line of curvature, an asymptotic line or a directrix curve of the Enneper Surface, the three points C, C_1, C_2 , which correspond to P in every case trace a plane curve.

We have studied in this paper the metrical properties of certain geometrical configurations which have been built up in connection with a point on a surface. We have made a study of these configurations for all the points of the Surface of Enneper to which the developments apply and have found the following properties of this surface:

a) *The axis of the cylindroid is always perpendicular to the XY -plane, with the middle point of the congruence always in this plane. Consequently, the axes of the osculating linear complexes of the asymptotic lines and the two directrices are always parallel to the XY -plane.*

b) *The axes of these complexes intersect and are perpendicular to each other. Their point of intersection is the middle point of the congruence and consequently they are in the XY -plane.*

c) *The surface contains two plane curves, the intersections of the surface by the XZ - and YZ -planes, at all of whose points, excepting the origin, the*

directrices of the first and second kind are perpendicular to each other, the axis of the cylindroid in each case lying in the plane of the curve. For all these points the parameters of the complexes are numerically equal and opposite in sign.

d) There are four points for which the directrix of the second kind coincides with the normal to the surface. Two of these points are upon each of the curves in the XZ- and YZ-planes and are the maximum and minimum points on the oval of these curves with reference to the axes of symmetry.

By means of the consideration of this special Surface of Enneper we have verified the results of the paper and have established the existence of real points of all of the special types that have been defined; points for which the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other, other points for which these axes intersect while the directrices of the two kinds are perpendicular to each other, and points for which the directrix of the second kind coincides with the normal to the surface. In the absence of any general existence theorems the fact that such points are shown to exist has an important bearing.

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